Granada Summer School
Quantum Matter - Foundations and Applications
September 15 - 19, 2013

UNIVERSITY OF INNSBRUCK

Non-Equilibrium Physics with Open Many-Body Quantum Systems

IQOQI
AUSTRIAN ACADEMY OF SCIENCES

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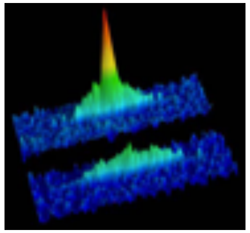
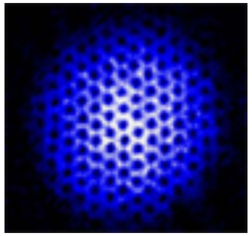
SFB
Coherent Control of Quantum Systems

FWF Der Wissenschaftsfonds.

Motivation

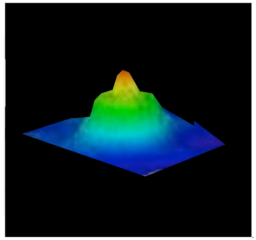


Bose-Einstein Condensate
(1995)



Vortices
(1999)

Many-body physics
with cold atoms



Mott Insulator
(2002)

Fermion superfluid
(2003)

Common theme:

many-body
system

Temperature T,
particle number N

- closed system (isolated from environment)
- stationary states in thermodynamic equilibrium

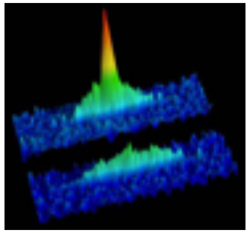
➡ thermalization/equilibration (PennState, Berkeley, Chicago, ...)

➡ sweep and quench many-body dynamics (Munich, Vienna)

➡ metastable excited many-body states (Innsbruck, MIT, ...)

➡ ...

Motivation



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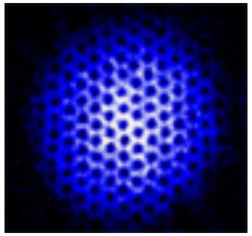

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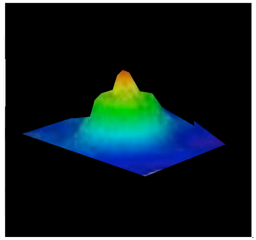

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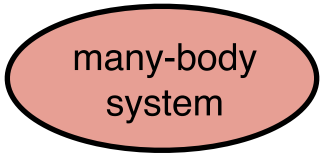
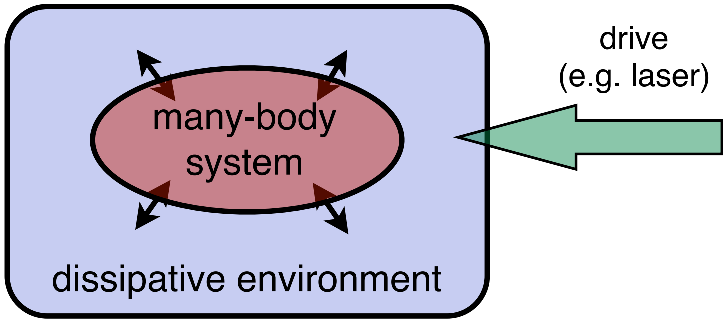


Temperature T,
particle number N

- closed system (isolated from environment)
- stationary states in thermodynamic equilibrium

Novel Situation: Cold atoms as open many-body systems



- natural occurrences of dissipation

➡ no immediate condensed matter counterpart

- use manipulation tools of quantum optics

➡ drive/dissipation as dominant resource of many-body dynamics!

➡ defines non-equilibrium situation in many-body stationary state

Plan of the Lecture

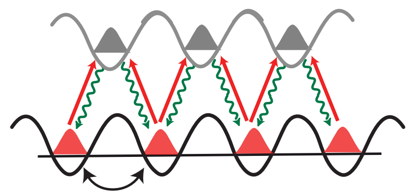
- Open system character on various length scales:

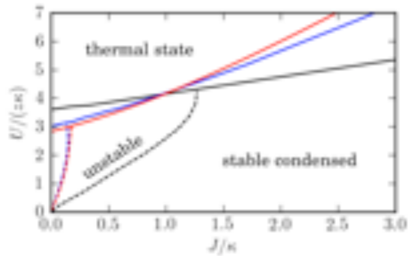
microscopic	thermodynamic	long wavelength
quantum optics	many-body physics	statistical mechanics

Part I: Dissipation Engineering and Many-Body Physics in Open Atomic Systems

- Open quantum systems
- Dissipation engineering in many-body systems
- Non-equilibrium phase transitions from competing unitary and dissipative dynamics

$$\partial_t \rho = -i[H,\rho] + \mathcal{L}[\rho]$$

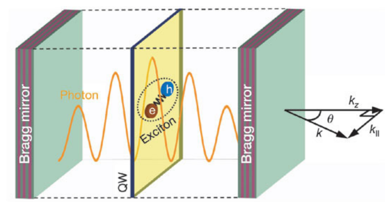


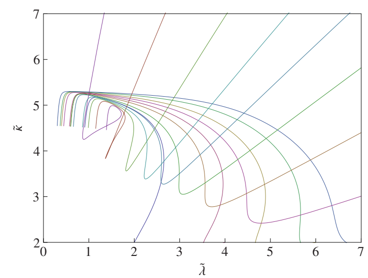


Part II: Many-Body Physics and Statistical Mechanics in Open Systems with Natural Dissipation

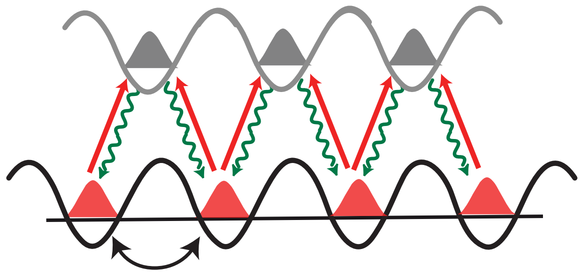
- Keldysh functional integral for open systems
- Experimental platforms and microscopic models
- Critical behavior and universality
- Dynamical criticality in driven-open systems

$$\mathrm{e}^{\mathrm{i}\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi \mathrm{e}^{\mathrm{i}S_M[\Phi+\delta\Phi]}$$





Part I: Dissipation Engineering and Many-Body Physics in Open Atomic Systems



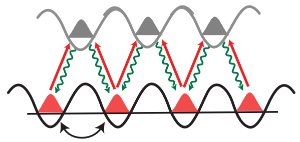
Outline

- Open quantum systems

$$\partial_t \rho = -i[H_S, \rho] + \kappa \sum_\alpha J_\alpha \rho J_\alpha^\dagger - \tfrac{1}{2}\{J_\alpha^\dagger J_\alpha, \rho\}$$

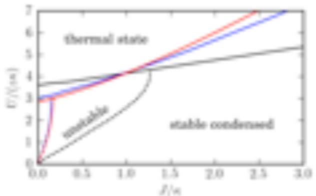
- scale separations in quantum optics
- quantum master equations

- Dissipation engineering in many-body systems



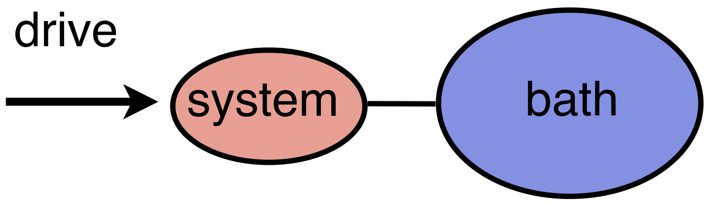
- dark states
- driven dissipative BEC

- Competing unitary and dissipative dynamics



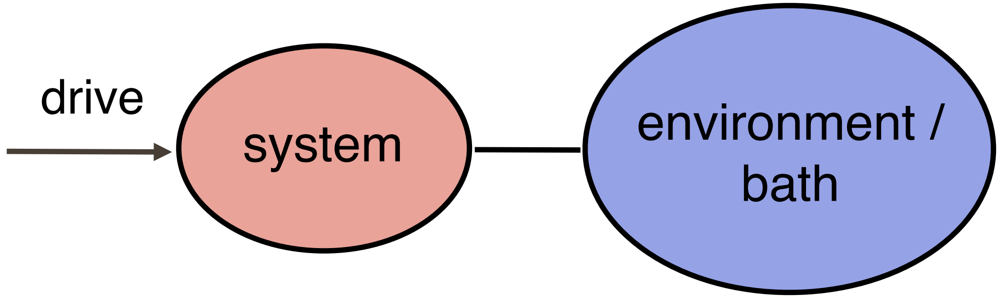
- dynamical phase transition
- non-equilibrium phase diagram

Brief Reminder:
Open Quantum Systems



Open Quantum Systems

$$H = H_{\rm S} + H_B + H_{\rm int}$$



$$H_S \sim \omega_0$$

typical scale

$$H_B = \int d\omega \, \omega b_\omega^\dagger b_\omega$$

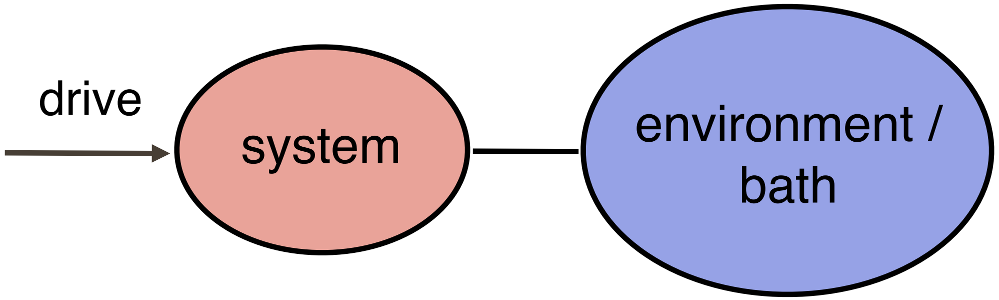
continuum bath of harmonic oscillators

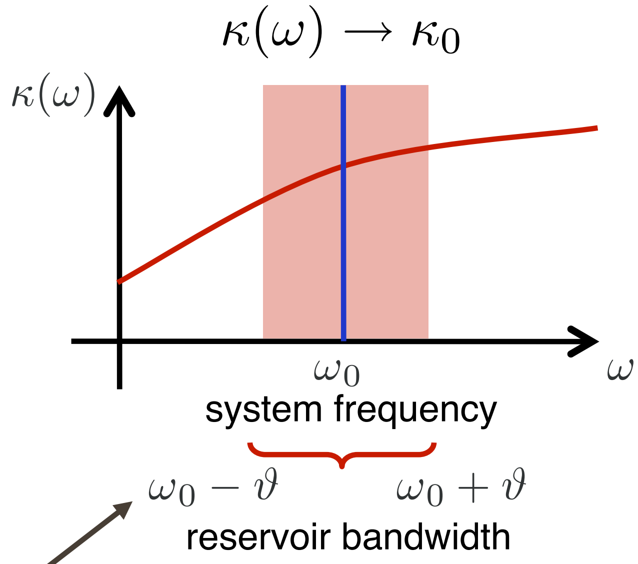
$$H_{\rm int} = i \int d\omega \kappa(\omega) \left[b_\omega^\dagger J - b_\omega J^\dagger \right]$$

quantum jump / Lindblad operators
polynomial in system operators

linear bath operator coupling to the system

Open Quantum Systems





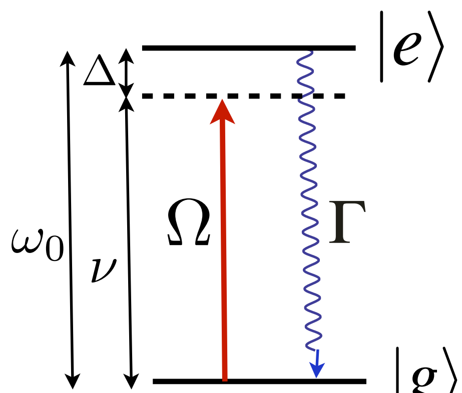
Three approximations:

(1) Born approximation: $\kappa(\omega)/\omega_0 \ll 1$

(2) Markov approximation: $\kappa(\omega) \approx$ const. $\Rightarrow \kappa(t-t') \sim \delta(t-t')$

(3) Rotating wave approximation: $\dfrac{\omega_0 - \nu}{\omega_0 + \nu} \ll 1$

driven system $\qquad \omega_0 - \nu = \Delta \qquad$ detuning



in this example:

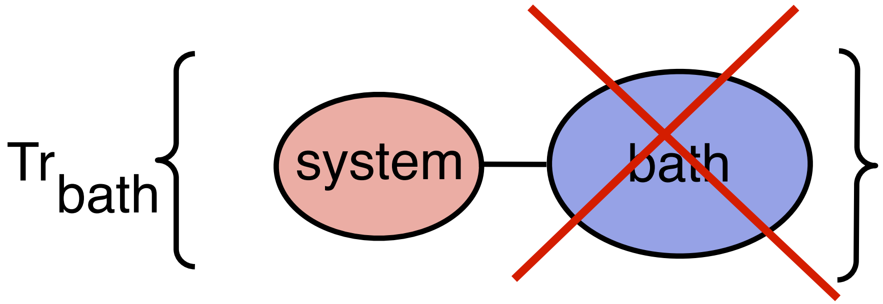
system Hamiltonian $\quad H = (|e\rangle, |g\rangle) \begin{pmatrix} \Delta & \Omega \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} \langle e| \\ \langle g| \end{pmatrix}$

jump operator $\quad J_\alpha = |g\rangle\langle e| = \sigma^-$

Quantum Master Equation

$$\partial_t \rho_{\text{tot}} = -i[H_S + H_B + H_{\text{int}}, \rho_{\text{tot}}]$$

➡ Eliminate bath degrees of freedom in second order time-dependent perturbation theory



effective system dynamics from Master Equation (zero temperature bath)

Lindblad quantum jump operators

$$\boxed{\partial_t \rho = -i[H_S, \rho] + \underbrace{\kappa \sum_\alpha J_\alpha \rho J_\alpha^\dagger - \tfrac{1}{2}\{J_\alpha^\dagger J_\alpha, \rho\}}_{\mathcal{L}[\rho]}}$$

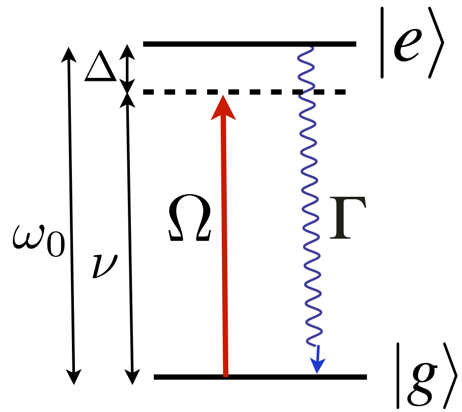
$\mathcal{L}[\rho]$ Liouvillian operator in Lindblad form

- Structure: second order perturbation theory
- mnemonic: norm conservation $\partial_t \mathrm{tr}\rho = 0$

Open Quantum Systems as Driven Systems

- Most (all?) of the non-equilibrium features to be discussed root in the driven nature of quantum optical systems

- Consider two-level system:
  - without drive, upper level inaccessible
  - drive / pump means to put in large amount of energy. Does not happen "spontaneously"
  - large scale separation: bath may look as zero temperature reservoir though it is not (cf. radiation field)



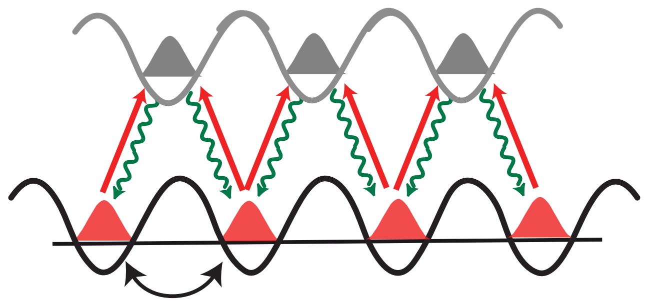
- Implications:
  - no obedience of the second law of thermodynamics (state purification)
  - independent unitary and dissipative dynamics (different physical origins)
  - no guarantee for detailed balance, once unitary and dissipative dynamics compete
  - NB: contrasts equilibrium: relaxational (dissipative) and reversible (coherent) dynamics have the same origin (Hamiltonian)

Part I

Part II

➡ such conditions may be achieved in many-body systems as well (though not generic)

Driven Dissipative BEC



Formulation of the Goal

- Devise purely dissipative evolution which drives into desired pure state

$$\partial_t \rho = -i\cancel{[H_S, \rho]} + \kappa \sum_\alpha J_\alpha \rho J_\alpha^\dagger - \tfrac{1}{2}\{J_\alpha^\dagger J_\alpha, \rho\}$$

$$\boxed{\rho(t) \xrightarrow{t\to\infty} \rho_{ss} \stackrel{!}{=} |D\rangle\langle D|}$$

pure state (“dark state”)

mixed state
typically

$$|D\rangle = |BEC\rangle$$

first example

- Contrast this to standard thermodynamic equilibrium scenario:

$$\rho \sim e^{-H/k_B T} \xrightarrow{T\to 0} |E_g\rangle\langle E_g|$$

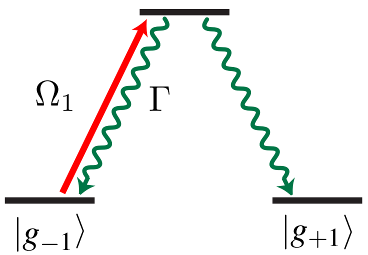
cooling to ground state by coupling to a zero temperature reservoir

Dark States in Quantum Optics

- Goal: pure BEC as steady state solution, independent of initial density matrix:

$$\rho(t) \longrightarrow |BEC\rangle\langle BEC| \text{ for } t \rightarrow \infty$$

- Such situation is well-known quantum optics (three level system): optical pumping
(Kastler, Aspect, Cohen-Tannoudji; Kasevich, Chu; ...)



$$\rho(t) \xrightarrow{t\to\infty} |g_+\rangle\langle g_+|$$

➡ Driven dissipative dynamics “purifies” the state

➡ $|g_+\rangle$ is a “dark state” decoupled from dissipative evolution

➡ More generally: dark state is a dissipative zero mode (time evolution stops)

$$J_\alpha|D\rangle = 0 \quad \forall\alpha$$

$$\mathcal{L} = \kappa\sum_\alpha J_\alpha\rho J_\alpha^\dagger - \tfrac{1}{2}\{J_\alpha^\dagger J_\alpha, \rho\}$$

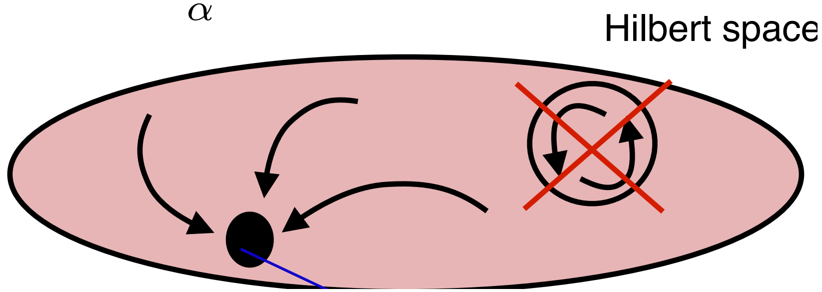
- Interesting situation: unique dark state solution

➡ dissipation increases purity

$$\boxed{\partial_t \,\mathrm{tr}(\rho^2) < 0}$$

➡ directed motion in Hilbert space

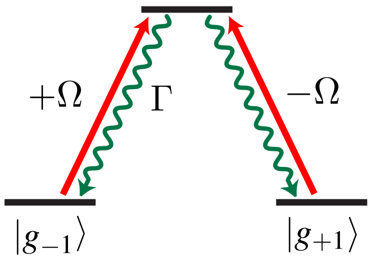
$$\rho \xrightarrow{t\to\infty} |D\rangle\langle D|$$

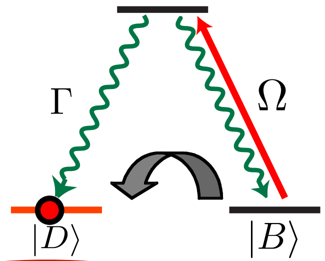


Dark states: An analogy

SD et al. Nat. Phys. (2008)
F. Verstraete et al. Nat. Phys. (2009)

- optical pumping: three internal (electronic) levels (Aspect, Cohen-Tannoudji; Kasevich, Chu)

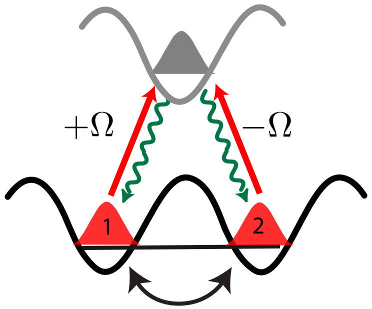


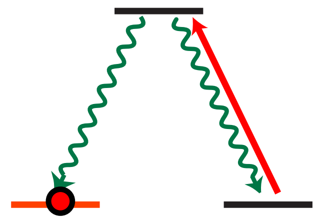


dark state — bright state

$$|D\rangle \sim |g_{+1}\rangle + |g_{-1}\rangle \qquad |B\rangle \sim |g_{+1}\rangle - |g_{-1}\rangle$$

- 1 atom on 2 sites: external (spatial) degrees of freedom

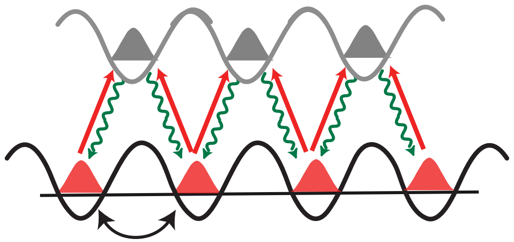




$$(a_1^\dagger + a_2^\dagger)\,|\mathrm{vac}\rangle \qquad (a_1^\dagger - a_2^\dagger)\,|\mathrm{vac}\rangle$$

symmetric — anti-symmetric

- N atoms on M sites

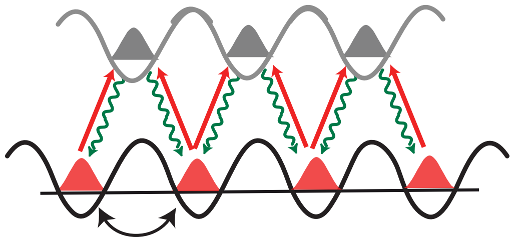


$$|BEC\rangle = \frac{1}{N!}\Big(\sum_\ell a_\ell^\dagger\Big)^N |vac\rangle$$

➡ drive and dissipation: many-particle optical pumping into

Driven Dissipative lattice BEC

$$\partial_t \rho = \kappa \sum_\alpha J_\alpha \rho J_\alpha^\dagger - \tfrac{1}{2}\{J_\alpha^\dagger J_\alpha, \rho\}$$



- Consider jump operator (1D):

$$J_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1})$$

- Interpretation: any antisymmetric component of a particle's superpositon on i, i+1 mapped onto the symmetric one

(1) BEC state is a dark state:

$$|BEC\rangle = \frac{1}{N!}\Big(\sum_\ell a_\ell^\dagger\Big)^N |vac\rangle$$

$$J_i|\mathrm{BEC}\rangle = 0 \;\forall i \qquad [(a_i - a_{i+1}), \sum_\ell a_\ell^\dagger] = 0$$

(2) BEC state is the only dark state:

- $(a_i^\dagger + a_{i+1}^\dagger)$ has no eigenvalues (on N-1 particle Hilbert space)
- $(a_i - a_{i+1})$ has unique zero eigenvalue

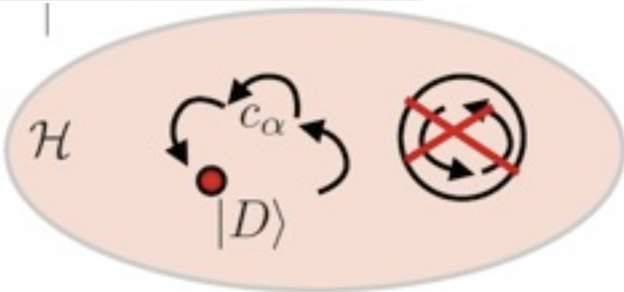
$$(a_i - a_{i+1}) \;\forall i \longrightarrow (1 - e^{\mathrm{i}q})a_q \;\forall q$$

Driven Dissipative lattice BEC

(3) Uniqueness: IBEC> is the only stationary state (sufficient condition)

If there exists no subspace of the full Hilbert space which is left invariant under the set $\{J_\alpha\}$, then the only stationary states are the dark states

(4) Compatibility of unitary and dissipative dynamics



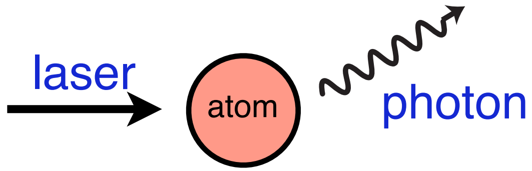
$|D\rangle$ be an eigenstate of H, $H\,|D\rangle = E\,|D\rangle$

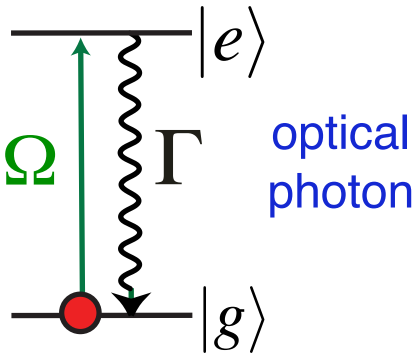
$$\rho(t) \xrightarrow{t\to\infty} |D\rangle\,\langle D|$$

- Long range order in many-body system from quasi-local dissipative operations
- Uniqueness: Final state independent of initial density matrix
- Criteria are general: jump operators for AKLT states (spin model), d-wave and topological states (fermions)

Physical Realization: Reservoir Engineering

- driven two-level atom + spontaneous emission

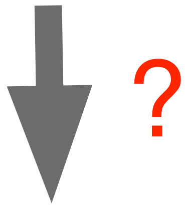




- coherent drive: optical laser light
- reservoir: vacuum modes of the radiation field (T=0)

$$\omega \sim 2\pi \times 10^{14} Hz$$

Quantum optics ideas/techniques

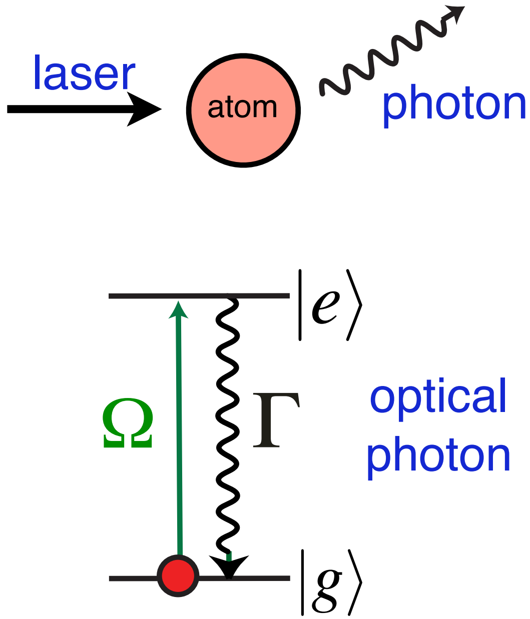


(many body) cold atom systems

- much lower energy scales...

Physical Realization: Reservoir Engineering

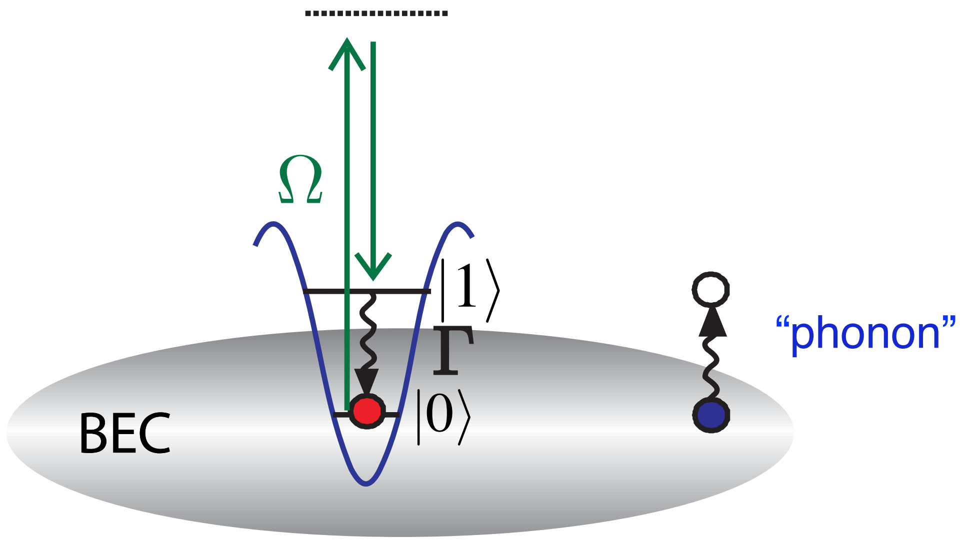
- driven two-level atom + spontaneous emission



- coherent drive: optical laser light
- reservoir: vacuum modes of the radiation field (T=0)

$$\omega \sim 2\pi \times 10^{14} Hz$$

- trapped atom in a BEC reservoir



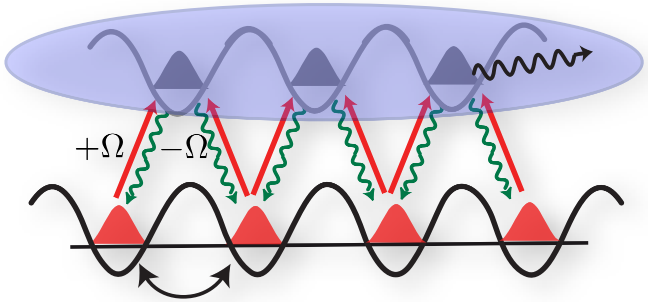
- coherent drive: Raman laser
- reservoir: Bogoliubov excitations of the BEC (at temperature T)

$$\omega_{bd} \sim 2\pi \times kHz$$

Physical Realization: Reservoir Engineering

- Idea: immersion of coherently driven lattice system into BEC reservoir

target setting



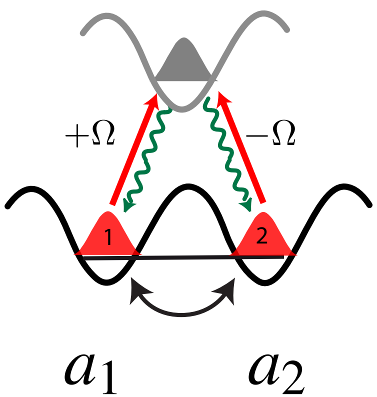
jump operators

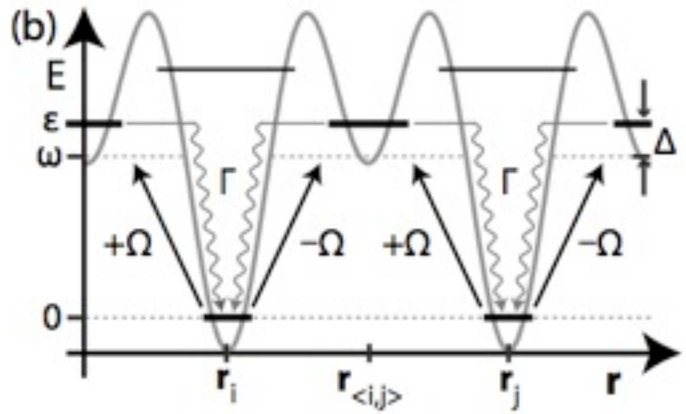
$$\boxed{J_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1})}$$

- geometric lattice setup: Λ-type level structure via optical superlattice

auxiliary system

system of interest





Physical Realization: Reservoir Engineering

- Idea: immersion of coherently driven lattice system into BEC reservoir

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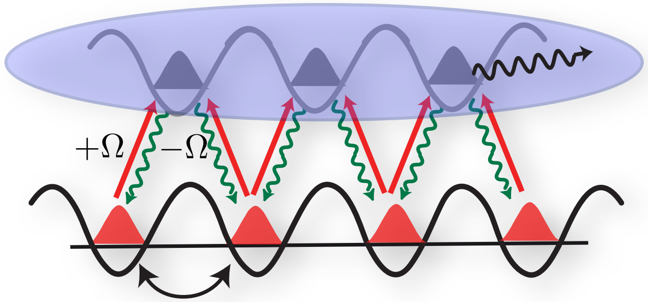
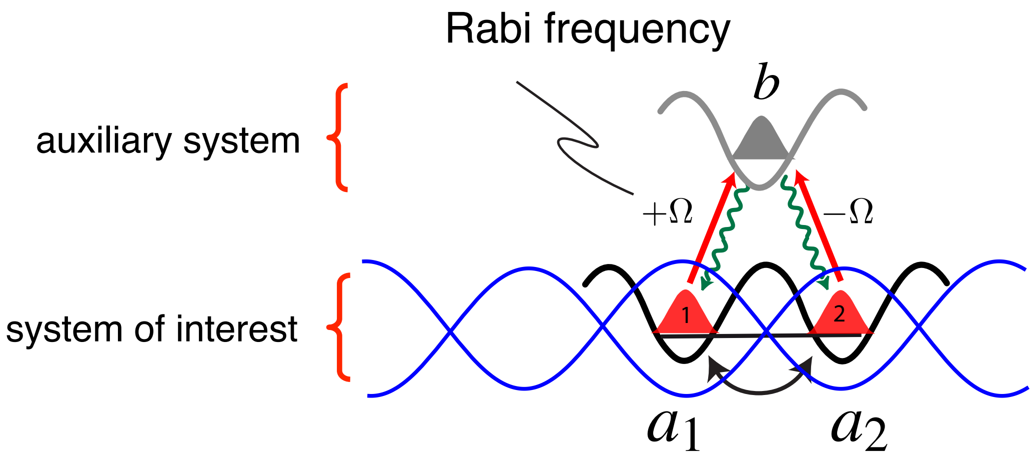


described jump operators

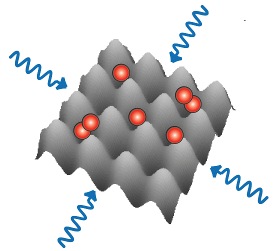
$$\boxed{J_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1})}$$

(i) Drive: coherent coupling to auxiliary system with double wavelength Raman laser



$$\boxed{\lambda_{\text{laser}} = 2\lambda_{\text{lattice}}}$$

driving laser



pairwise antisymmetric drive

$$\Omega_1 b^\dagger a_1 + \Omega_2 b^\dagger a_2 + h.c.$$

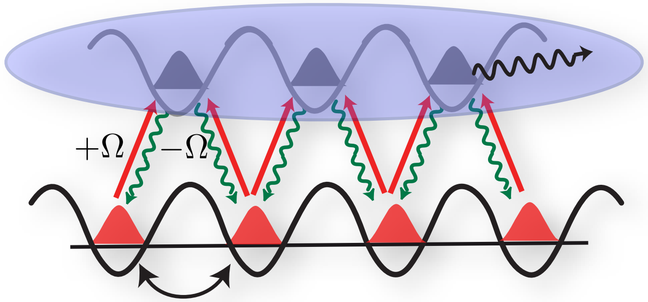
$$= \Omega b^\dagger (a_1 - a_2) + h.c.$$

for $\quad \Omega_2 = e^{i\pi}\Omega_1 = -\Omega_1$

Physical Realization: Reservoir Engineering

- Idea: immersion of coherently driven lattice system into BEC reservoir

target setting



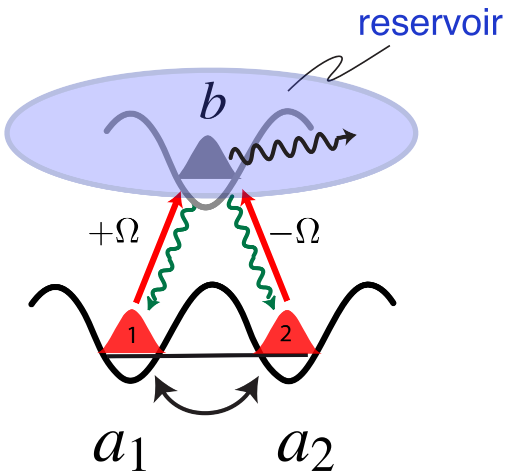
described jump operators

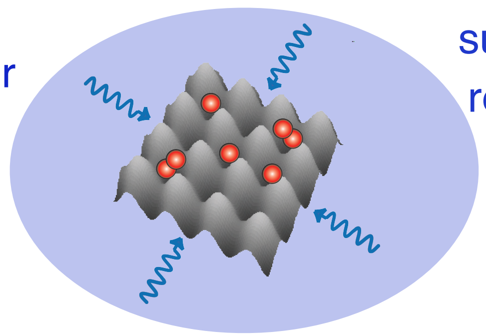
$$\boxed{J_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1})}$$

(ii) Dissipation: phonon emission into superfluid reservoir

auxiliary system

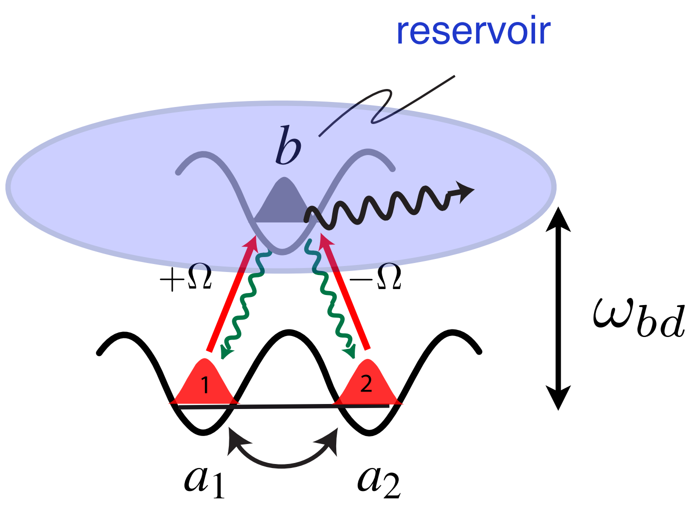
system of interest

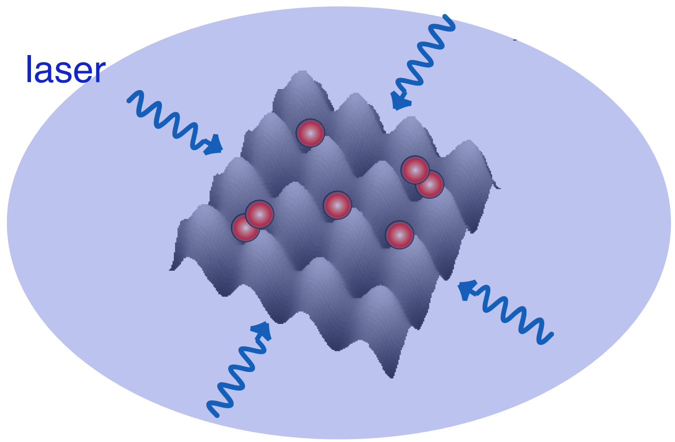




- microscopically: s-wave interaction of system and bath particles
- BEC in bath gives standard qo system-bath setting

Cooling into BEC with another BEC?





- band separation ω_{bd} largest energy scale in the problem

- reservoir BEC = reservoir of Bogoliubov phonon excitations
- has temperature T_{BEC}

$$T_{BEC} \ll \omega_{bd}$$

bath occupation

$$\Rightarrow n\left(\frac{\omega_{bd}}{T_{BEC}}\right) \ll 1$$

➡ effective zero temperature reservoir

➡ can reach system entropies well below bath (possible due to pumping, cf. fridge)

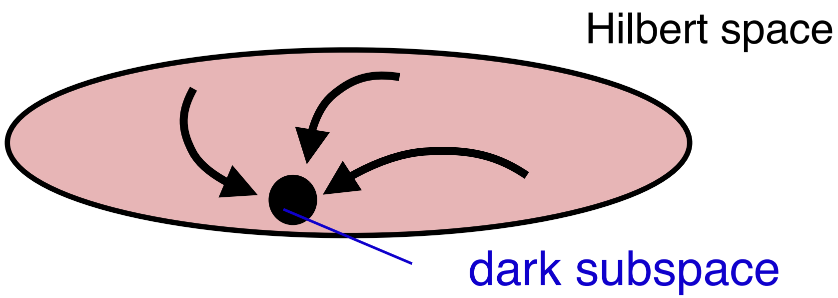
Physical Realization

Summary:

- Long range phase coherence from quasi-local dissipative operations
- - Coherent drive: locks phases

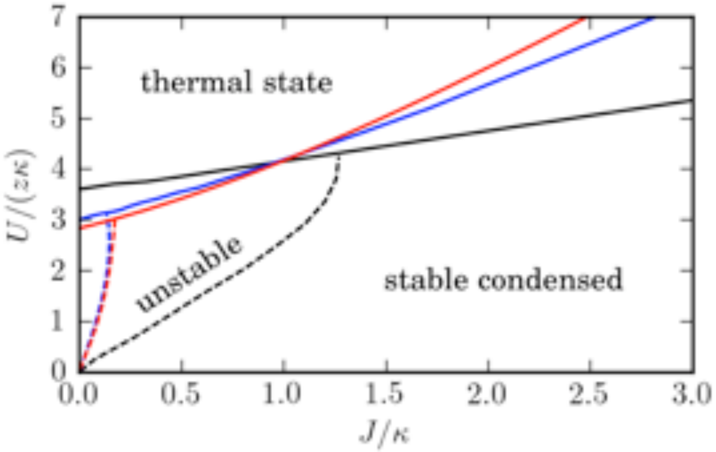
  - Dissipation: randomizes

  - Conspiracy: directed motion in Hilbert space, purification



- The coherence of the driving laser is mapped on the matter system
- Setting is therefore robust (commensurability condition on driving and lattice laser)

Competition of Unitary vs. Dissipative Dynamics



SD, A. Tomadin, A. Micheli, R. Fazio, P. Zoller, Phys. Rev. Lett. **105**, 015702 (2010);
A. Tomadin, SD, P. Zoller, Phys. Rev. A **108**, 013611 (2011).

Physical Picture: Nonequilibrium Phase Transition

- Nonequilibrium master equation evolution:

drives into BEC with rate κ

$$\frac{d\rho}{dt} = -i\,[H,\rho] + \mathcal{L}\rho$$

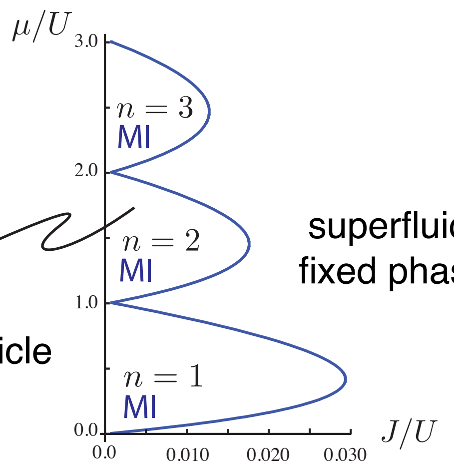
$$H = -J \sum_{<i,j>} a_i^\dagger a_j + U \sum_i a_i^{\dagger 2} a_i^2$$

Competition

- Compare to superfluid / Mott insulator quantum phase transition

  ➡ competition between kinetic and interaction energy

Mott lobes, quantized particle number



superfluid, fixed phase

Interference pattern

superfluid: Mott   back to superfluid

Reminder: Mott Insulator-Superfluid Phase Transition

$$H = -J \sum_{\langle i,j \rangle} b_i^\dagger b_j - \mu \sum_i \hat{n}_i + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1)$$

- Hopping J favors delocalization in real space:
- Condensate (local in momentum space!)
- Fixed condensate phase: Breaking of phase rotation symmetry

$$\langle b_i \rangle \sim e^{\mathrm{i}\varphi}$$

- Interaction U favors localization in real space for integer particle numbers:
- Mott state with quantized particle no.
- no expectation value: phase symmetry intact (unbroken)

➡ Competition gives rise to a quantum phase transition as a function of

$$U/J$$

Physical Picture: Nonequilibrium Phase Transition

- Nonequilibrium master equation evolution:

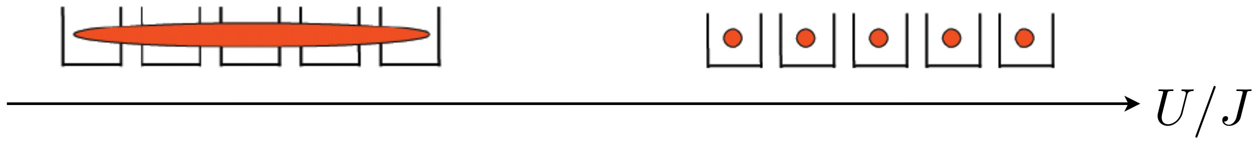
drives into BEC with rate κ

$$\frac{d\rho}{dt} = -i\,[H,\rho] + \mathcal{L}\rho$$

$$H = -J \sum_{<i,j>} a_i^\dagger a_j + U \sum_i a_i^{\dagger 2} a_i^2$$

Competition

- Analogy to superfluid / Mott insulator quantum phase transition



• enhancement of superfluidity:	kinetic energy J	driven dissipation κ
• suppression of superfluidity:	interaction U	interaction U

➡ Expect phase transition as function of J/U κ/U

➡ Question: What are the true analogies and differences to equilibrium (quantum) phase transitions?

Mixed State Gutzwiller Approach

- Argumentation must be based on equation of motion

- Strategy: approximation scheme interpolating between limiting cases

$\kappa \gg U$	$\kappa \ll U$
dissipative condensate	see below!

- Implementation: Gutzwiller product ansatz for the density operator

$$\rho(t) = \prod_i \rho_i(t)$$

  - onsite (quantum) fluctuations treated exactly
  - (connected) spatial correlations neglected
  - allows to describe mixed states (unlike zero temperature Gutzwiller)

➡ Nonlinear Mean Field Master Equation for reduced density operator

- We will additionally account for a finite hopping J

From Weak to Strong Coupling

- Weak interactions: dissipative Gross-Pitaevskii equation (coherent states)

$$\partial_t \psi_\ell = -i(-J \sum_{\langle \ell' | \ell \rangle} \psi_{\ell'} + U|\psi_\ell|^2 \psi_\ell) - 2\kappa \sum_{\langle \ell' | \ell \rangle} (\psi_\ell - \psi_{\ell'} + \psi^*_{\ell'} \psi^2_\ell - |\psi_{\ell'}|^2 \psi_{\ell'})$$

- Strong interaction destroys the phase coherence:

dephasing & average out

transformation to rotating frame $V \equiv e^{iU\hat{n}(\hat{n}-1)t}$

annihilation operator in rotating frame $V\hat{b}V^{-1} = e^{-iU\hat{n}t}\hat{b} = \sum_n e^{inUt} |n\rangle\langle n|\hat{b}$

➡ suppression of off-diagonal order

$\sim \psi$

at dark state

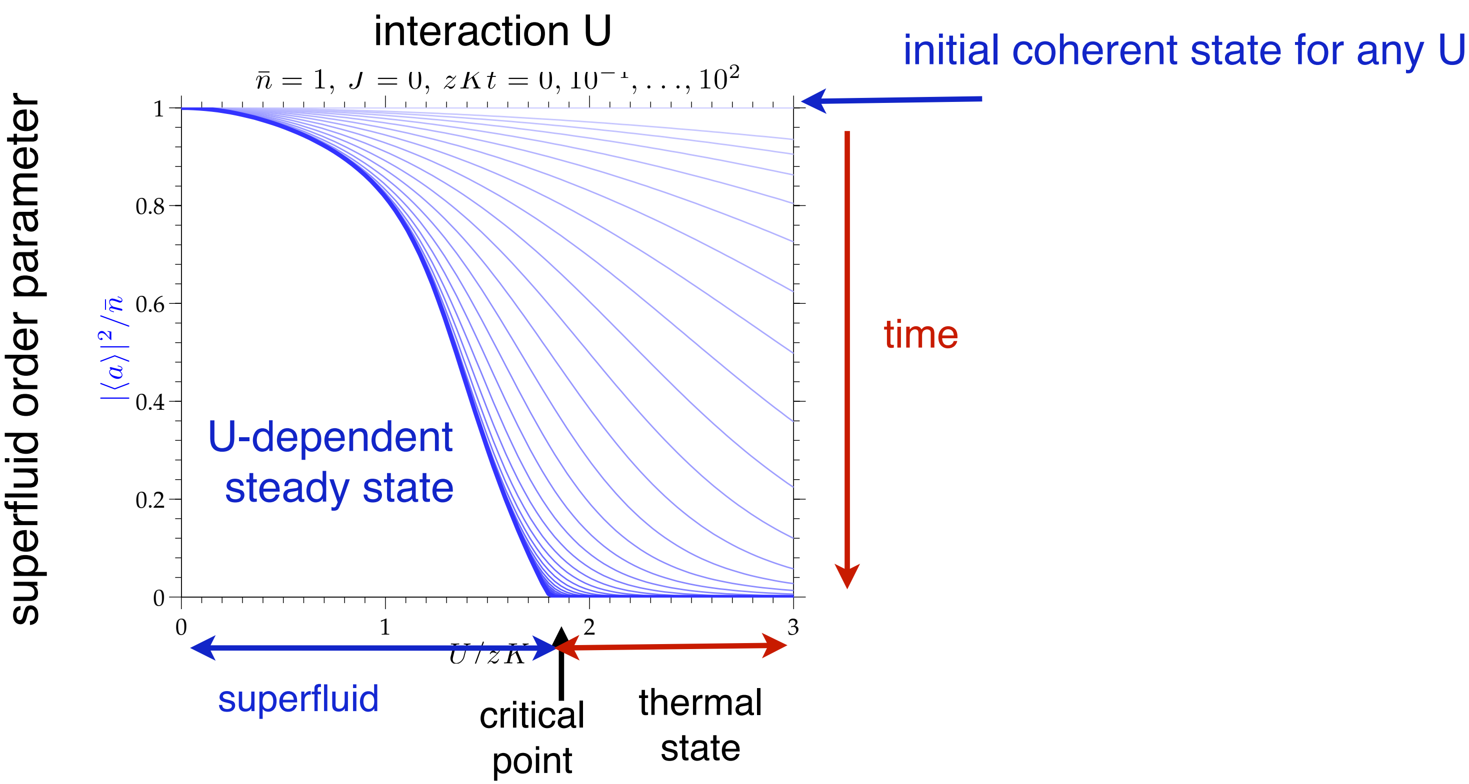
- Master equation reduces to

$$\partial_t \rho_\ell = \kappa[(\bar{n}+1)(2b_\ell \rho b_\ell^\dagger - \{b_\ell^\dagger b_\ell, \rho_\ell\}) + \bar{n}(2b_\ell^\dagger \rho_\ell b_\ell - \{b_\ell b_\ell^\dagger, \rho_\ell\})]$$

- Thermal equation with thermal (mixed) state solution

➡ the svstem acts as its own reservoir

Dependence of the Steady State on the Interaction



Nonequilibrium phase transition between pure and mixed state,
driven by a competition between unitary and dissipative dynamics

- Development in time of the non-analyticity at the critical point
- Shares features of:
  - Quantum phase transition: interaction driven
  - Classical phase transition: ordered phase terminates in a thermal state
- No signature of commensurability effects (Mott) due to strong mixing of U

Analytical Approach in the Limit of Low Density

- Many-body problem: relevant information in the low order correlation functions

- Study the equations of motion of the correlation functions

  $\{\langle (b_\ell^\dagger)^n b_\ell^m \rangle\}$ in principle: infinite and nonlocal hierarchy

- Introduce a power counting: $\boxed{b_\ell \sim \sqrt{n}, b_\ell^\dagger \sim \sqrt{n}}$

  and keep only the leading order for $n \to 0$

➡ Infinite hierarchy exhibits a closed nonlinear subset for low order correlation functions

Critical Exponent of the Phase Transition

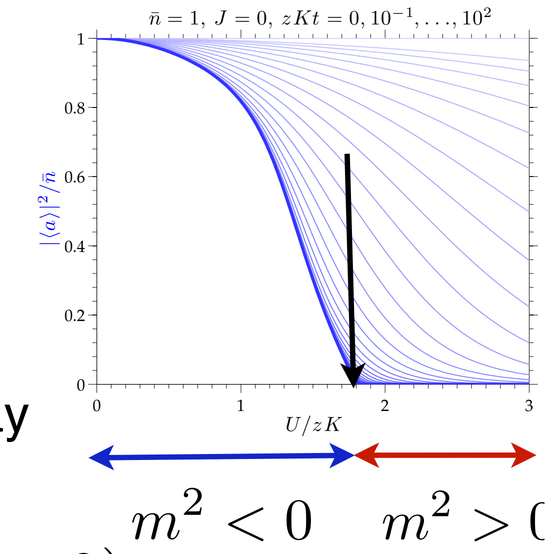
- Critical exponents can be extracted from approaching the phase transition in time

- Expect form of the order parameter evolution

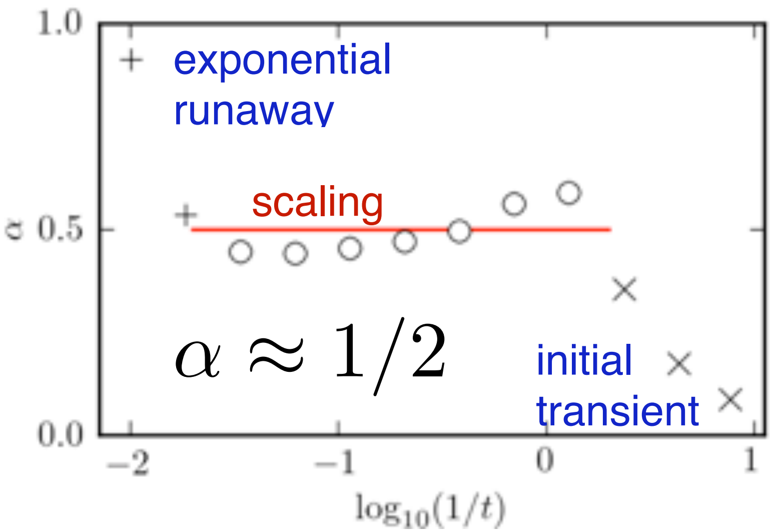
$$|\psi(t)| \sim \frac{e^{-m^2 t}}{t^{\alpha}}$$

real part of lowest eigenvalue: “mass”

- At criticality: zero eigenvalue and thus dominant polynomial decay



- Numerical Result (high density):



- Analytical Result $(n \to 0)$:

at criticality, Landau-Ginzburg type cubic but dissipative nonlinearity

$$\boxed{|\psi(t)| \sim t^{-1/2}, \quad \alpha = 1/2}$$

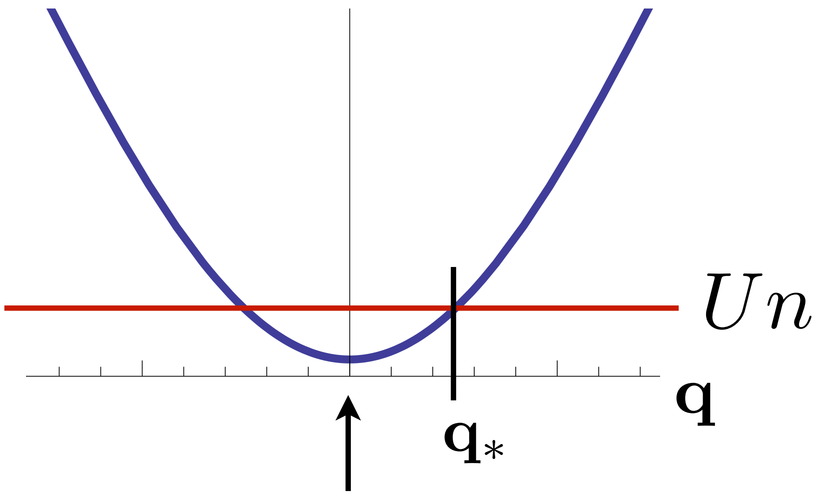
Mean field value as expected.
But governs the time evolution.

➡ Critical behavior could be studied experimentally from following the time evolution of

Order-order phase transition at weak coupling

- qualitative picture: weak damping in vicinity of dark state (linearized field equation)

$$\mathbf{q} \to 0 : \quad \kappa_{\mathbf{q}} \approx \kappa(2n+1)\mathbf{q}^2$$



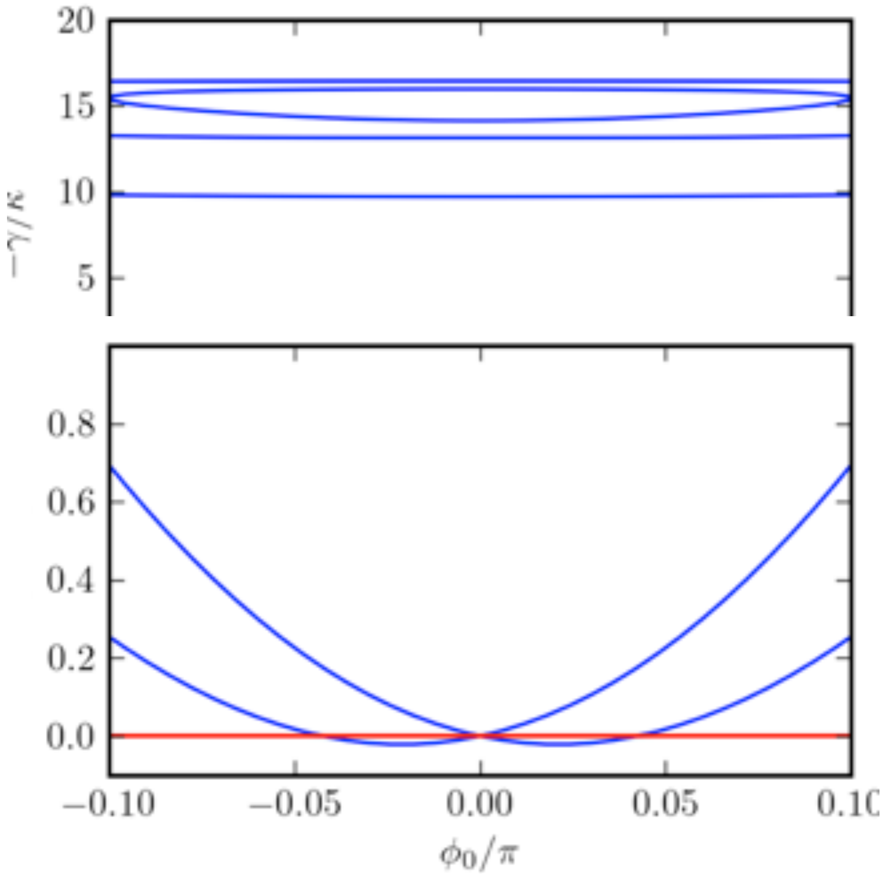
dark state at $\mathbf{q} = 0$

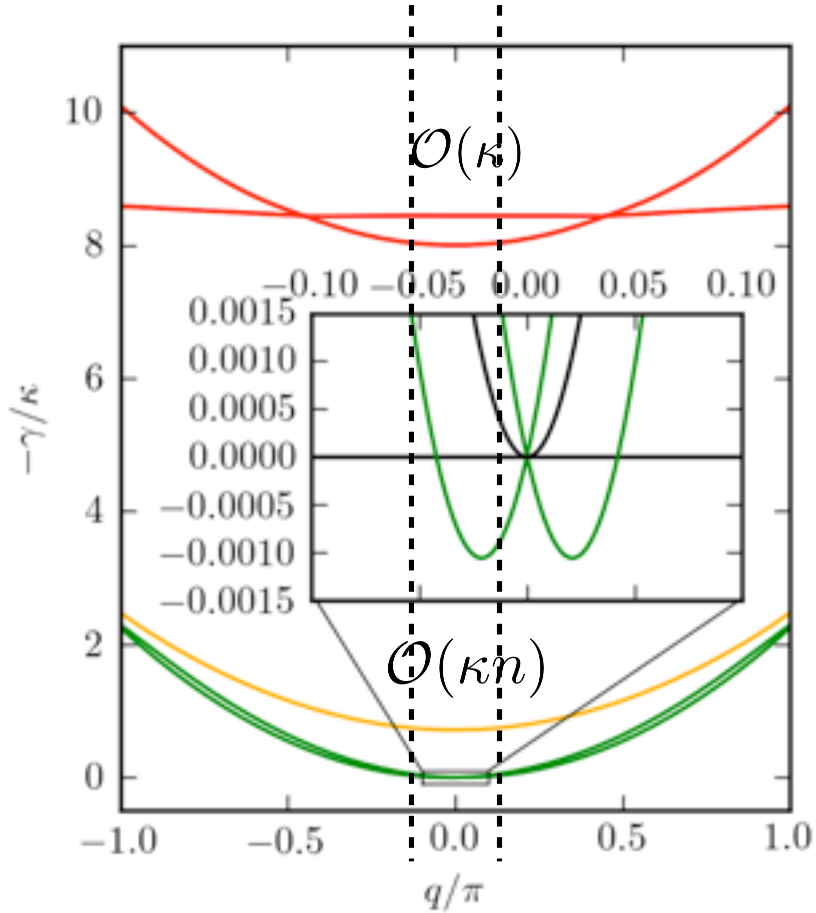
- the scale Un competes with hopping and dissipation
- there always is a $|\mathbf{q}_*|$ where the competition is of order unity

➡ can expect qualitative effects

Linear Response around Homogeneous State

- Imaginary part of the Liouvillian as function of quasimomentum, $J \ll \kappa$





100 sites, high densities, full mean field system

Infinite system, low densities, 7x7 linear system of EoMs

➡ Existence of dissipatively unstable modes is a universal feature of the regime $J \ll \kappa$

➡ low density limit: the unstable modes belong to single particle sector

Reduction to the Low-Lying Modes

- Adiabatic elimination of the fast-decaying modes (two times)

$$\left(\begin{array}{c}\partial_t\Psi_1\\ 0\equiv\partial_t\Psi_2\end{array}\right)=\left(\begin{array}{cc}M_{11} & M_{12}\\ M_{21} & M_{22}\end{array}\right)\left(\begin{array}{c}\Psi_1\\ \Psi_2\end{array}\right)\Bigg\}$$

collection of low density correlation functions

solve for the fast modes Ψ_2 and obtain slow modes equation only

- Low momentum equation of motion for of the condensate fluctuations only

$$\partial_t\left(\begin{array}{c}\Delta\psi_q\\ \Delta\psi^*_{-q}\end{array}\right)=\left(\begin{array}{cc}Un+\epsilon_{\mathbf{q}}-i\kappa_{\mathbf{q}} & Un+9un\kappa_{\mathbf{q}}\\ -Un-9un\kappa_{\mathbf{q}} & -Un-\epsilon_{\mathbf{q}}-i\kappa_{\mathbf{q}}\end{array}\right)\left(\begin{array}{c}\Delta\psi_q\\ \Delta\psi^*_{-q}\end{array}\right)$$

$$\epsilon_q\equiv Jq^2,\quad \kappa_q=2(2n+1)\kappa q^2$$

bare hopping at low momentum

bare dissipative rate

➡ renormalization of the off-diagonal terms

➡ absent in the dissipative GPE

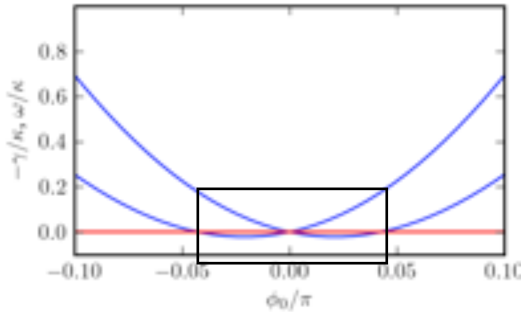
Origin of the Instability

renormalization correction

- Complex spectrum of the low-lying single particle excitations:

$$\gamma_{\mathbf{q}} = \kappa_{\mathbf{q}} + ic|\mathbf{q}|, \quad c = \sqrt{2Un(J - 9Un/(2z))}$$

- Interpretation: Below a critical value

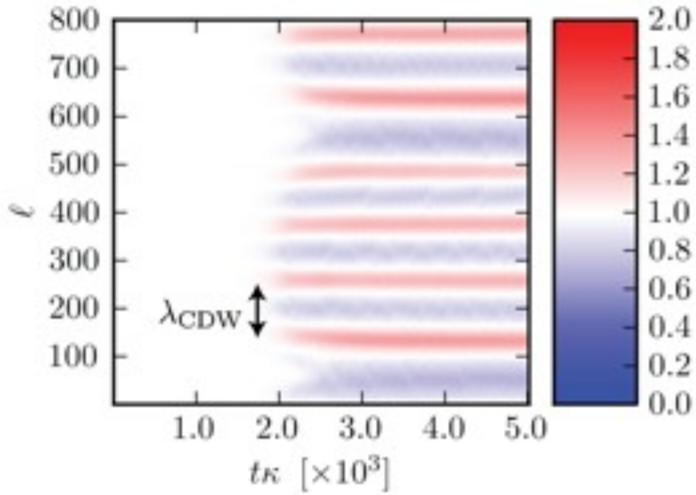


$$\boxed{J = 9Un/(2z)}$$

the speed of sound becomes imaginary.

This term always dominates at sufficiently small momenta. Its sign is opposite to $\kappa_{\mathbf{q}}$

- The fate of the system beyond linear response:

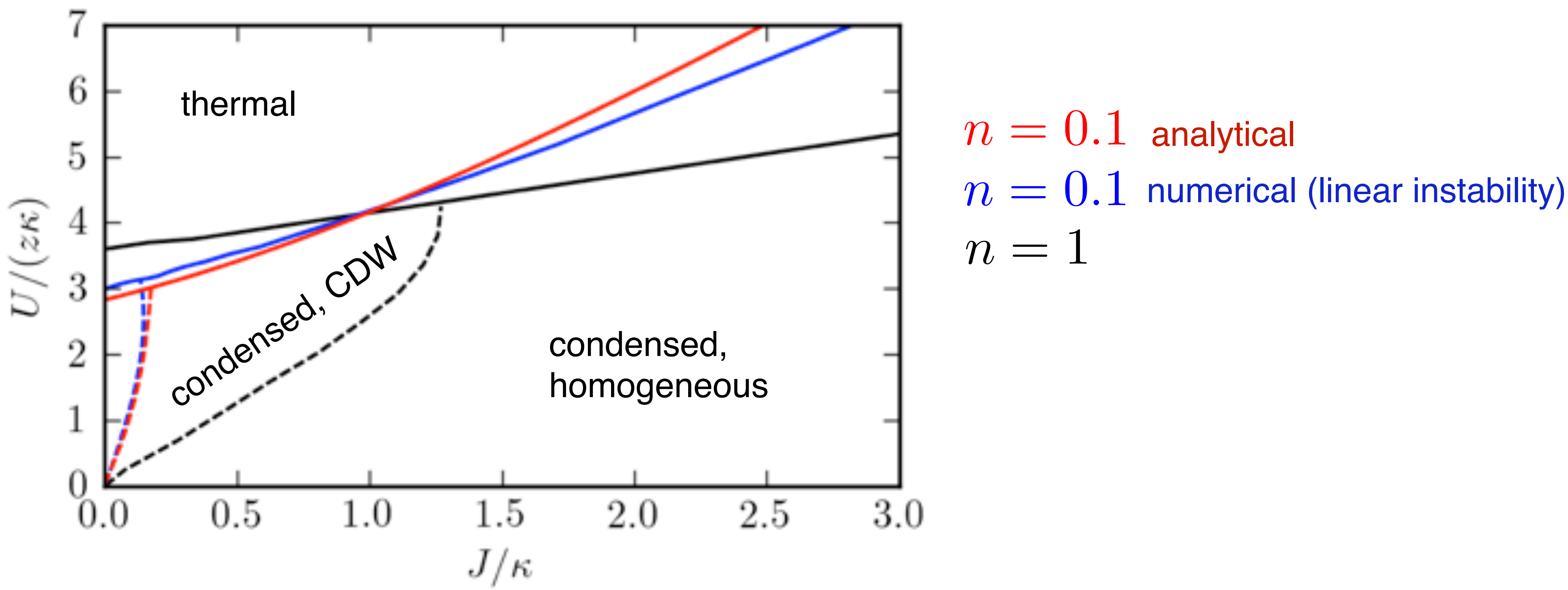


density profile signature: spontaneous breaking of translation symmetry

maximum instability momentum transmuted into CDW wavelength

➡ The dynamical instability is fluctuation induced, a weak coupling phenomenon, and an

The Steady State Phase Diagram



- Strong coupling second order phase transition to a thermal-like disordered state
- Homogeneous dissipative condensate is unstable against CDW order for infinitesimal interaction
- Condensed phase and homogeneous condensate can be stabilized by finite coherent hopping

Summary and further aspects

By merging techniques from quantum optics and many-body systems:

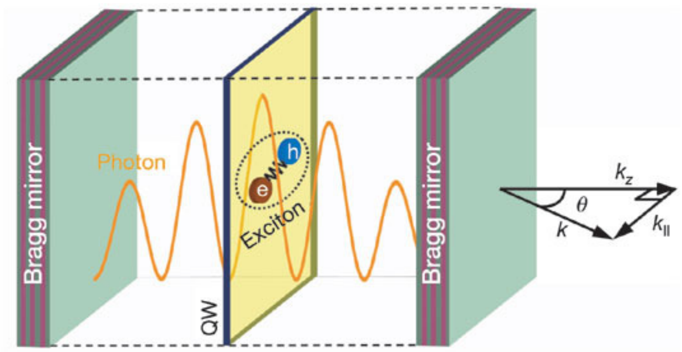
Driven dissipation can be used as controllable tool in cold atom systems.

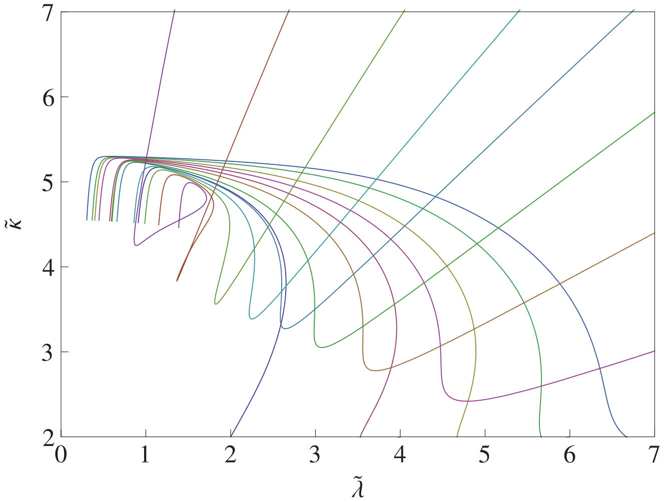
- Pure states with long range correlations from quasilocal dissipation
- New many-body physics: Nonequilibrium phase transition driven via competition of unitary and dissipative dynamics

---

- Additional physical platforms for dissipation engineering: trapped ions, microcavity arrays
- Bosons: What is the nature / universality class of the dynamical phase transition?
- Fermions: dissipative pairing and targeting of topological states of matter

Part II:
Many-Body Physics and Statistical Mechanics in open systems with natural dissipation





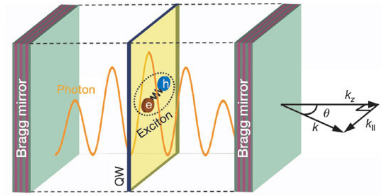
Outline

- Keldysh functional integral for open systems

$$e^{i\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi e^{iS_M[\Phi+\delta\Phi]}$$

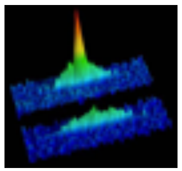
- mapping quantum master equations to functional integrals
- responses and correlations

- Experimental platforms and microscopic models



- microscopic derivation for stochastic exciton-polariton models
- symmetries and low momentum dynamics

- Critical behavior and universality

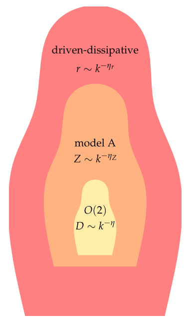


$\simeq$

- reminder: criticality in equilibrium
- universality

- Dynamical criticality in driven-open systems

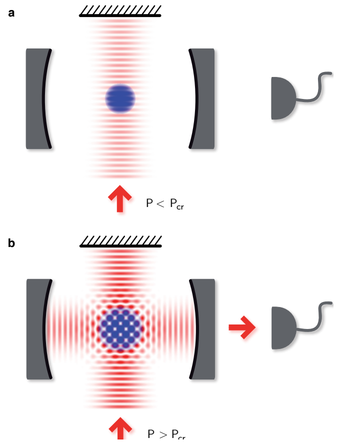
- Key Questions:
  - universality out of equilibrium?
  - relation to equilibrium criticality?
  - Thermalization, decoherence?



Motivation: Driven-dissipative many-body dynamics

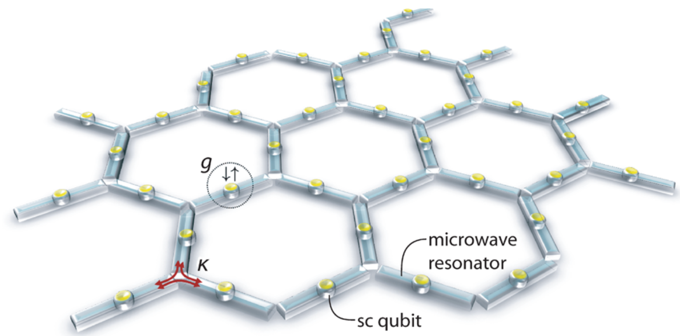
- experimental systems on the interface of quantum optics and many-body physics

  - Driven-open Dicke models



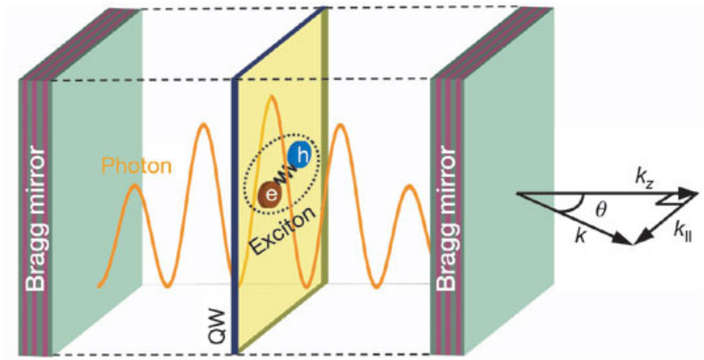
Baumann et al., Nature 2010

  - Coupled microcavity arrays: driven open Hubbard models



Koch et al., PRA 2010

  - exciton-polariton systems in semiconductor quantum wells



Kasprzak et al., Nature 2006

- other platforms (light-matter):
  - ➡ polar molecules
  - ➡ optical Feshbach resonances
  - ➡ trapped ions
  - ➡ nanomechanics

Keldysh Functional Integral for Open Systems

$$e^{i\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi e^{iS_M[\Phi+\delta\Phi]}$$

Why working with Functional Integrals?

- Feynman's formulation of quantum mechanics

REVIEWS OF

MODERN PHYSICS

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

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Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

IT is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems

- Advantages of the functional formulation of quantum field theory

- general:
  - unified language from quantum dots to quantum gravity
  - powerful techniques: diagrammatic perturbation theory; collective variables; renormalization group

- non-equilibrium Keldysh
  - closer to the real-time formulations of quantum mechanics
  - yields directly observable quantities (responses and correlations)
  - indispensable for non-Hamiltonian systems:
    - disorder
    - dissipation

    infinite harmonic baths!
  - open the powerful toolbox of quantum field theory for many-body non-equilibrium situations

Basic Idea: Keldysh functional integral

- Compare: $$U(t,t_0) = e^{-iH(t-t_0)}$$
  - Schroedinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \qquad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

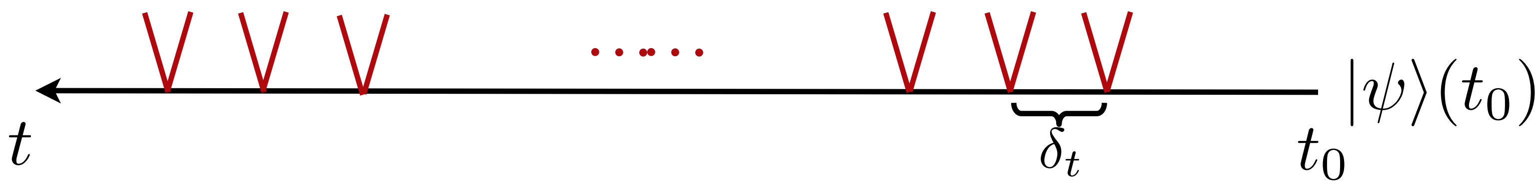
  - Heisenberg equation: evolving a state (density) matrix

$$\partial_t\rho(t) = -i[H,\rho(t)] \qquad \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

    - identical for pure (factorizable) states $\rho = |\psi\rangle\langle\psi|$

- First case: functional integral via “Trotterization” of time interval and insertion of coherent states:

$$e^{iH(t-t_0)} = \lim_{N\to\infty}\left(1 + i\delta_t H\right)^N \qquad \delta_t = \frac{t-t_0}{N}$$



➡ single set of degrees of freedom for vector evolution

➡ analogous procedure for thermal equilibrium: formal analogy of evolution operator e^{-iHt} and “imaginary time evolution operator” $\rho_{\rm eq} = e^{-\beta H}$

Basic Idea: Keldysh functional integral

- Compare:

$$U(t,t_0) = e^{-iH(t-t_0)}$$

  - Schroedinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

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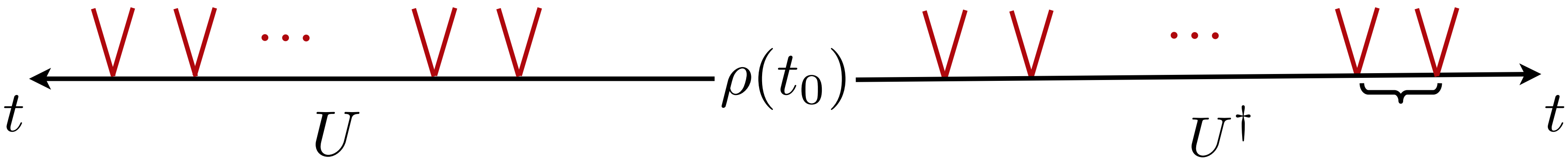
$$\partial_t\rho(t) = -i[H,\rho(t)] \quad \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

    - identical for pure (factorizable) states $\rho = |\psi\rangle\langle\psi|$

- Second case: “Trotterization” on both sides:

$$e^{iH(t-t_0)} = \lim_{N\to\infty}(1+i\delta_t H)^N \qquad \delta_t = \frac{t-t_0}{N}$$



➡ two sets of degrees of freedom for matrix evolution

Basic Idea: Keldysh functional integral

- Compare:

$$U(t,t_0) = e^{-iH(t-t_0)}$$

  - Schroedinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

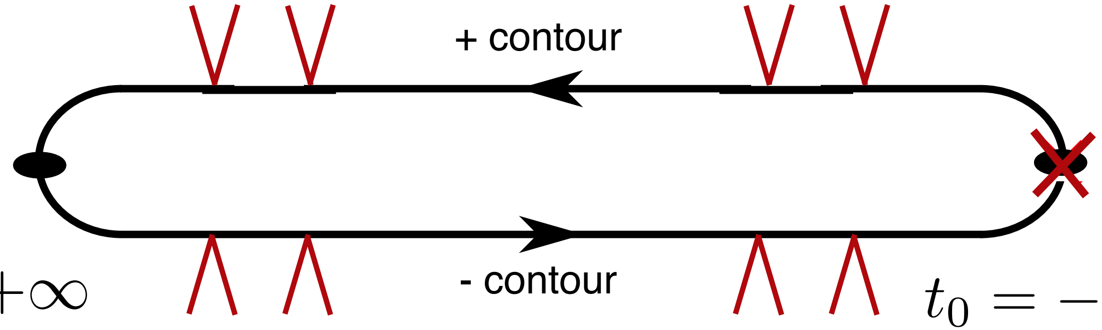
  - Heisenberg equation: evolving a state (density) matrix

$$\partial_t\rho(t) = -i[H,\rho(t)] \quad \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

    - identical for pure (factorizable) states $\rho = |\psi\rangle\langle\psi|$

- Finally, we are interested in a “partition function”

$$Z = \mathrm{tr}\rho(t) = \mathrm{tr}\rho(t_0) = 1$$



$t_f = +\infty$ $\qquad$ $t_0 = -\infty$

➡ the trace contracts the evolution times

➡ information on all stages: $t_0 \to -\infty, t_f \to +\infty$

Implementation: Keldysh integral for quantum master equations

- Goal: Functional integral representation of “partition function” for the quantum master equation

$$\partial_t \rho = \mathcal{L}\ \rho = -i\,[H,\rho] + \sum_\alpha \kappa_\alpha \left(2L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}\right)$$

- i.e. representation in the basis of coherent states of

$$Z = \mathrm{tr}\rho(t)$$

- Step 1: formal solution of the master equation
  - master equation not “separable” (action of L_α from both sides simultaneously)
  - but linear in the density matrix: solution with “superoperator”

$$\rho(t) = e^{(t-t_0)\mathcal{L}}\ \rho_0 \overset{\text{def}}{=} \lim_{N\to\infty} \left(1+\delta_t \mathcal{L}\right)^N \rho_0 \qquad \delta_t = \frac{t-t_0}{N}$$

➡ unravelling/meaning in terms of concatenated infinitesimal time steps

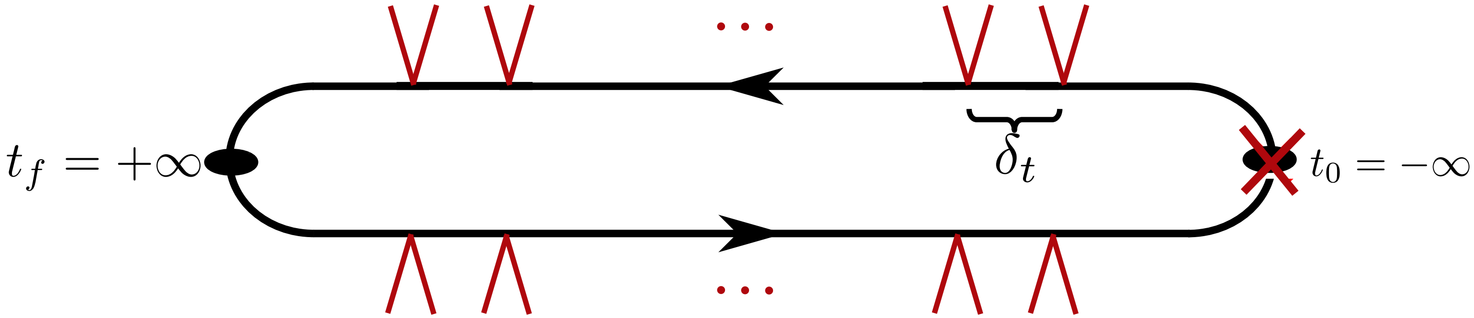
➡ in each of them, apply rhs of the master equation

Implementation: Keldysh integral for quantum master equations

- partition function:

$$Z(t) = \mathrm{Tr}\left(\rho(t)\right) = \mathrm{Tr}\left(\left[\lim_{N\to\infty}\prod_{l=1}^{N}\left(1+\delta_t^{(l)}\mathcal{L}\right)\right]\rho_0\right) = 1$$

- now insert coherent states after each time step:



- Reminder:
  - Coherent states (bosons) – eigenstates to the annihilation operators a_i:
  - properties:

$$a_i|\phi\rangle = \phi_i|\phi\rangle,\ \langle\phi|a_i^\dagger = \langle\phi|\phi_i^*$$

$$|\phi\rangle = e^{\sum_i \phi_i a_i^\dagger}|\mathrm{vac}\rangle \qquad \text{explicit form}$$

$$\langle\theta|\phi\rangle = e^{\sum_i \theta_i^*\phi_i},\ \langle\phi|\phi\rangle = e^{\sum_i \phi_i^*\phi_i} \qquad \text{overlap and normalization}$$

$$\mathbf{1}_{\mathrm{Fock}} = \int\prod_i \frac{d\phi_i^* d\phi_i}{\pi} e^{-\sum_i \phi_i^*\phi_i}|\phi\rangle\langle\phi| \qquad \text{completeness}$$

  - Note: The creation operators do not have eigenstates

Implementation: Keldysh integral for quantum master equations

- partition function:

$$Z(t) = \mathrm{Tr}\left(\rho(t)\right) = \mathrm{Tr}\left(\left[\lim_{N\to\infty}\prod_{l=1}^{N}\left(1+\delta_t^{(l)}\mathcal{L}\right)\right]\rho_0\right) = 1$$

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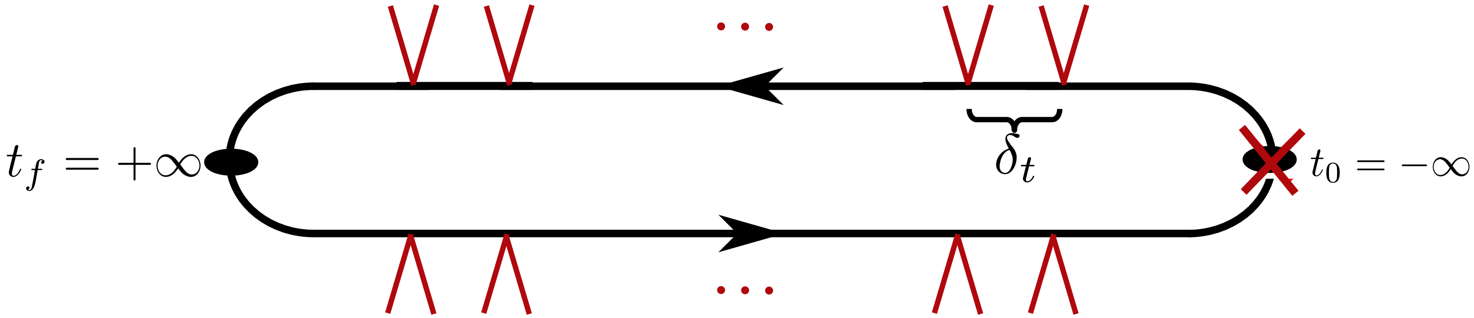


- mathematically:

$$Z(t) = \mathrm{Tr}\left(\underbrace{}_{\mathbf{1}_{N+}}\cdots\left[\left(1+\delta_t^{(2)}\mathcal{L}\right)\left(\underbrace{}_{\mathbf{1}_{1+}}\left[\left(1+\delta_t^{(1)}\mathcal{L}\right)\left(\underbrace{}_{\mathbf{1}_{0+}}\rho_0\underbrace{}_{\mathbf{1}_{0-}}\right)\right]\underbrace{}_{\mathbf{1}_{1-}}\right)\right]\cdots\underbrace{}_{\mathbf{1}_{N-}}\right)$$

- evaluate step l:

$$|\phi_{l+}\rangle\langle\phi_{l+}|\left[\left(1+\delta_t^{(l)}\mathcal{L}\right)\left(|\phi_{l-1+}\rangle\langle\phi_{l-1+}|\ldots|\phi_{l-1-}\rangle\langle\phi_{l-1-}|\right)\right]|\phi_{l-}\rangle\langle\phi_{l-}|$$

Implementation: Keldysh integral for quantum master equations

- evaluate step l:

$$|\phi_{l+}\rangle\langle\phi_{l+}| \left[\left(1+\delta_t^{(l)}\mathcal{L}\right)\left(|\phi_{l-1+}\rangle\langle\phi_{l-1+}|\dots|\phi_{l-1-}\rangle\langle\phi_{l-1-}|\right)\right]|\phi_{l-}\rangle\langle\phi_{l-}|$$

$$= |\phi_{l+}\rangle\langle\phi_{l+}|\phi_{l-1+}\rangle\langle\phi_{l-1+}| \dots|\phi_{l-1-}\rangle\langle\phi_{l-1-}|\phi_{l-}\rangle\langle\phi_{l-}|$$

factor 1

$$+\frac{\delta_t^{(l)}}{i}|\phi_{l+}\rangle\langle\phi_{l+}|H|\phi_{l-1+}\rangle\langle\phi_{l-1+}| \dots|\phi_{l-1-}\rangle\langle\phi_{l-1-}|\phi_{l-}\rangle\langle\phi_{l-}|$$

$$-\frac{\delta_t^{(l)}}{i}|\phi_{l+}\rangle\langle\phi_{l+}|\phi_{l-1+}\rangle\langle\phi_{l-1+}| \dots|\phi_{l-1-}\rangle\langle\phi_{l-1-}|H|\phi_{l-}\rangle\langle\phi_{l-}|$$

Heisenberg commutator

$$-\delta_t^{(l)}\sum_\alpha \kappa_\alpha|\phi_{l+}\rangle\langle\phi_{l+}|L_\alpha^\dagger L_\alpha|\phi_{l-1+}\rangle\langle\phi_{l-1+}| \dots|\phi_{l-1-}\rangle\langle\phi_{l-1-}|\phi_{l-}\rangle\langle\phi_{l-}|$$

$$-\delta_t^{(l)}\sum_\alpha \kappa_\alpha|\phi_{l+}\rangle\langle\phi_{l+}|\phi_{l-1+}\rangle\langle\phi_{l-1+}| \dots|\phi_{l-1-}\rangle\langle\phi_{l-1-}|L_\alpha^\dagger L_\alpha|\phi_{l-}\rangle\langle\phi_{l-}|$$

dissipation

(anticommutator term)

$$+\delta_t^{(l)}\sum_\alpha 2\kappa_\alpha|\phi_{l+}\rangle\langle\phi_{l+}|L_\alpha|\phi_{l-1+}\rangle\langle\phi_{l-1+}| \dots|\phi_{l-1-}\rangle\langle\phi_{l-1-}|L_\alpha^\dagger|\phi_{l-}\rangle\langle\phi_{l-}|.$$

fluctuation

(quantum jump term)

- for normally ordered operators $H, L_\alpha^\dagger L_\alpha, L_\alpha, L_\alpha^\dagger$

  each matrix element can be computed, e.g.

$$|\phi_{l+}\rangle\langle\phi_{l+}|H(b^\dagger, b\ )|\phi_{l-1+}\rangle\langle\phi_{l-1+}| = H(\phi_{l+}^*, \phi_{l-1+})\ \ |\phi_{l+}\rangle\langle\phi_{l+}|\phi_{l-1+}\rangle\langle\phi_{l-1+}|.$$

➡ time-independent operator valued Liouvillian --->
time(l)-dependent complex valued Liouvillian functional

Implementation: Keldysh integral for quantum master equations

- time(l)-dependent complex valued Liouvillian functional

$$\mathcal{L}(\phi^*_{l+}, \phi_{l-1+}, \phi^*_{l-}, \phi_{l-1-})$$

$$= \frac{1}{i}\left(H(\phi^*_{l+}, \phi_{l-1+}) - H(\phi^*_{l-}, \phi_{l-1-})\right)$$

$$- \sum_\alpha \kappa_\alpha \left(\left(L^\dagger_\alpha L_\alpha\right)(\phi^*_{l+}, \phi_{l-1+}) + \left(L^\dagger_\alpha L_\alpha\right)(\phi^*_{l-}, \phi_{l-1-})\right) + \sum_\alpha 2\kappa_\alpha L_\alpha(\phi^*_{l+}, \phi_{l-1+}) L^\dagger_\alpha(\phi^*_{l-}, \phi_{l-1-}).$$

- factor 1: remember the completeness relation and overlaps $\mathbf{1} = \int \prod_i \frac{d\phi^*_i d\phi_i}{\pi} e^{-\phi^*_i \phi_i} |\phi\rangle\langle\phi|$

$$e^{-\phi^*_{l+}\phi_{l+}} |\phi_{l+}\rangle\langle\phi_{l+}|\phi_{l-1+}\rangle\langle\phi_{l-1+}| \ldots |\phi_{l-1-}\rangle\langle\phi_{l-1-}|\phi_{l-}\rangle\langle\phi_{l-}| e^{-\phi^*_{l-}\phi_{l-}}$$

$$= e^{-\phi^*_{l+}(\phi_{l+}-\phi_{l-1+})} \; |\phi_{l+}\rangle\langle\phi_{l-1+}| \ldots |\phi_{l-1-}\rangle\langle\phi_{l-}| \; e^{-(\phi^*_{l-}-\phi^*_{l-1-})\phi_{l-}}$$

$$= e^{i\delta_t^{(l)}\phi^*_{l+} i\partial_t \phi_{l+}} \; |\phi_{l+}\rangle\langle\phi_{l-1+}| \ldots |\phi_{l-1-}\rangle\langle\phi_{l-}| \; e^{-i\delta_t^{(l)}\phi^*_{l-} i\partial_t \phi_{l-}},$$

➡ gives rise to time evolution on the contour

- last step: take the continuum limit in time graining,

$$\delta_t^{(l)} \to 0, N \to \infty$$

Markovian dissipative action on the contour

- Markovian dissipative action

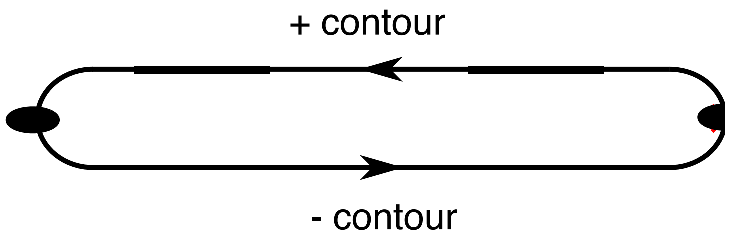
$$S = \int_{t_0}^{t_f} dt \, \left(\phi_+^*(t) i\partial_t \phi_+(t) - \phi_-^*(t) i\partial_t \phi_-(t) - i\mathcal{L}(\phi_+^*(t), \phi_+(t), \phi_-^*(t), \phi_-(t))\right).$$

$$\mathcal{L} = -i\left(H_+ - H_-\right) - \sum_\alpha \kappa_\alpha \left(2L_{\alpha,+}L^\dagger_{\alpha,-} - L^\dagger_{\alpha,+}L_{\alpha,+} - L^\dagger_{\alpha,-}L_{\alpha,-}\right)$$

$$H_\pm = H(\phi^*_\pm, \phi_\pm) \text{ etc.}$$

  - ➡ recognize Lindblad structure
  - ➡ simple translation table (for normal ordered Liouvillian)

• operator right of density matrix -> - contour
• operator left of density matrix -> + contour



- Functional integral representation of partition function

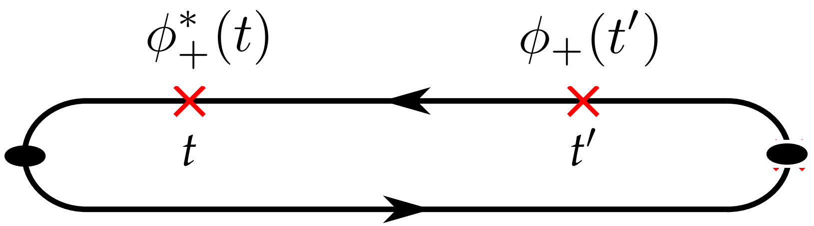
$$Z = \mathrm{tr}\rho(t) = \int \mathcal{D}[\Phi_+, \Phi_-] \;\; e^{iS[\Phi_+,\Phi_-]} = 1. \qquad \Phi_\pm = (\phi^*_\pm, \phi_\pm)^T$$

product of individual measures
in each time step

  - ➡ the partition function expresses conservation of probability
  - ➡ no direct physical information (unlike equilibrium: $\log Z \sim$ free energy)
  - ➡ physical information is in the correlation functions

Physical observables

- correlation functions: field insertions on the contour



- compute them: introduce sources (cf. Stat Mech)

$$Z = \mathrm{Tr}(1 \cdot \rho) = \langle 1 \rangle$$

$$Z[j_+, j_-] = \langle e^{\mathrm{i} \int (j_+ \phi_+^* + j_- \phi_-^* + c.c.)} \rangle$$

$$Z[0,0] = \langle 1 \rangle = 1$$

normalization

- example

$$\langle \mathcal{T}_C [\hat{\phi}^\dagger(t) \hat{\phi}(t')] \rangle = \frac{\delta^2 Z[j_+, j_-]}{\delta j_+(t) \delta j_+^*(t')} \Big|_{j=0}$$

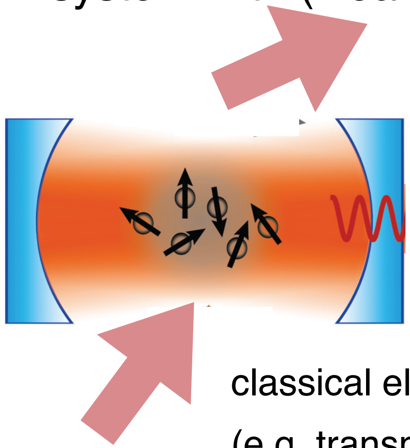
NB: Functional integrals always compute time-ordered correlation functions

Correlation vs. response functions

- two basic types of experiments:
  - correlation measurements: study without disturbing
  - response measurements: probe system with (weak) external fields

eg. quantum optics

study the photon output
(e.g. $g^{(2)}(\tau)$)



classical electromagnetic waves
(e.g. transmission/absorption experiments)

- directly delivered in the functional framework via basis transformation: “Keldysh rotation”

$$\begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_+ + \phi_- \\ \phi_+ - \phi_- \end{pmatrix}$$

“classical field”: center-of-mass coordinate
“quantum field”: relative coordinate

  - classical field can acquire finite expectation value (e.g. Bose condensation)
  - quantum / noise field cannot

Correlation vs. response functions

- the action written in this basis:

$$S = \int_{\omega,\mathbf{q}} (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + \text{interactions.}$$

➡ redundancy of the +/- basis eliminated (zero entry)

➡ the matrix is the inverse single particle Green's function:

- equation of motion (action principle):

$$\begin{pmatrix} \frac{\delta S}{\delta \phi_c^*} \\ \frac{\delta S}{\delta \phi_q^*} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix}}_{G^{-1}} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} \stackrel{!}{=} 0$$

(exact for free theory only)

- Green's function

$$G^{-1} \circ G = \mathbf{1}\delta(\omega - \omega')\delta(\mathbf{q} - \mathbf{q})$$

(Green's function diagonal in frequency/momentum space)

➡ single particle Green's function/propagator:

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} \qquad G^K = -G^R P^K G^A$$

Correlation vs. response: Interpretation by example

- master equation for decaying cavity:

$$\partial_t \rho = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger \hat{a}, \rho\})$$

- action:

$$S = \int dt (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i\kappa \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

time domain $a_\nu(t)$

$$= \int \frac{d\omega}{2\pi} (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \underbrace{\omega - \omega_0 + i\kappa}_{G^R(\omega)^{-1}} & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

frequency domain $a_\nu(\omega)$

- observables from the Green's functions:

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix}$$

$$G^K = -G^R P^K G^A$$

response

- Lorentzian spectral density $A(\omega) = \mathrm{Im} G^R(\omega) = \dfrac{2\kappa}{(\omega - \omega_0)^2 + \kappa^2}$

- decay of single-particle response: $G^R(t-t') = \int_\omega e^{i\omega(t-t')} G^R(\omega) = \theta(t-t') e^{i\omega(t-t')} e^{-\kappa(t-t')}$

correlation

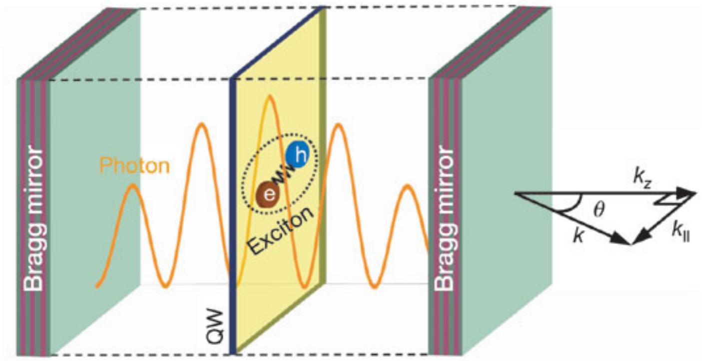
- cavity mode occupation in stationary state : $2\langle \hat{n}(t)\rangle + 1 = \langle \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}(t)\hat{a}^\dagger(t)\rangle = iG^K(t-t) = i\int_\omega e^{i\omega(t-t)} G^K(\omega) = 1 \quad (t \to \infty)$

$$\langle \hat{n}(t \to \infty)\rangle = 0$$

➡ correlation / statistical properties: G^K

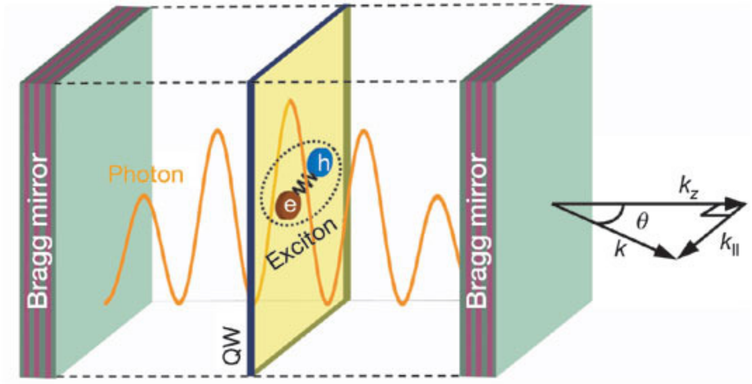
➡ response / spectral properties: G^R

Exciton-Polariton Condensates

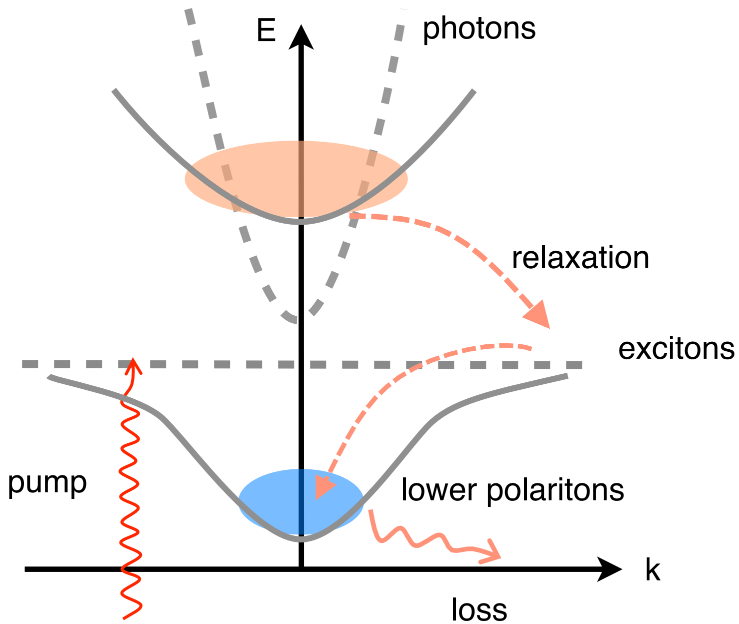


Exciton-polariton systems: qualitative picture

Kasprzak et al., Nature 2006



Imamoglu et al., PRA 1996



- experimental setup
- “level scheme”

- phenomenological description: stochastic driven-dissipative Gross-Pitaevskii-Eq

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\,|\phi|^2 \right] \phi + \zeta$$

propagation

pump & loss rates

elastic collisions

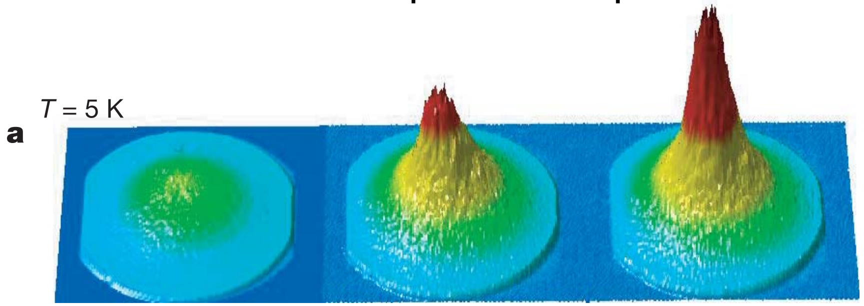
two-body loss

$$\langle \zeta^*(t,\mathbf{x})\zeta(t',\mathbf{x}')\rangle = \gamma\delta(t-t')\delta(\mathbf{x}-\mathbf{x}$$

mean field theory and non-universal aspects: Szymanska, Keeling, Littlewood PRL

Bose Condensation of Exciton-Polaritons

- Bose condensation seen despite non-equilibrium conditions



Kasprzak et al., Nature 2006

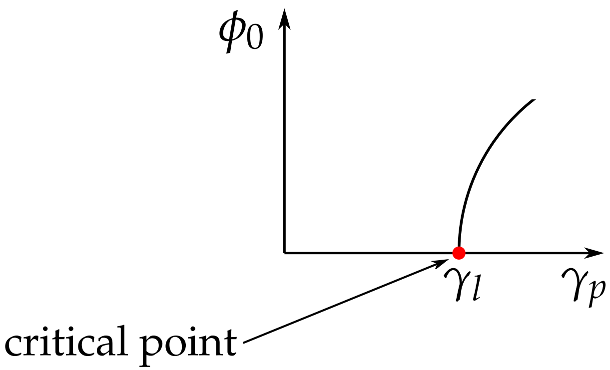
non-equilibrium stationary state:
balance of loss and gain

- stochastic driven-dissipative Gross-Pitaevskii-Eq

$$\cancel{i\partial_t \phi} = \left[\cancel{-\frac{\nabla^2}{2m}} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\,|\phi|^2 \right] \phi \cancel{+ \zeta}$$

mean field theory and non-universal aspects: Szymanska, Keeling, Littlewood PRL (04, 06); PR (07)); Wouters, Carusotto PRL (07,10)

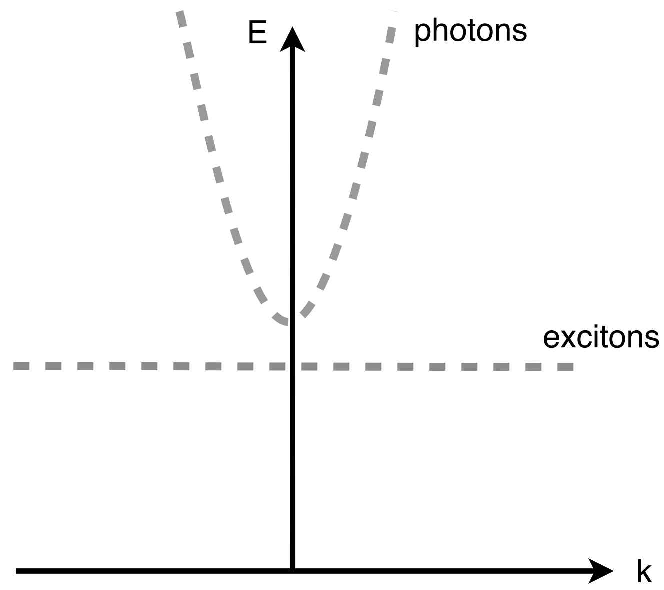
- mean field
  - neglect noise
  - homogeneous solution $\phi(\mathbf{x}, t) = \phi_0$



follows M. H. Szymanska, J. Keeling, P. B. Littlewood, arXiv:1206.1784

Microscopic Origin

- Starting point: coupled, open system of excitons and photons



  - Hamiltonian contribution:

$$H = H_{\rm ex} + H_{\rm ph} + H_{\rm int}$$

    - excitons: two-level fluctuators

$$H_{\rm ex} = \sum_j \epsilon_j \sigma_j^z = \sum_j \epsilon_j (\hat{d}_j^\dagger \hat{d}_j - \hat{c}_j^\dagger \hat{c}_j)$$

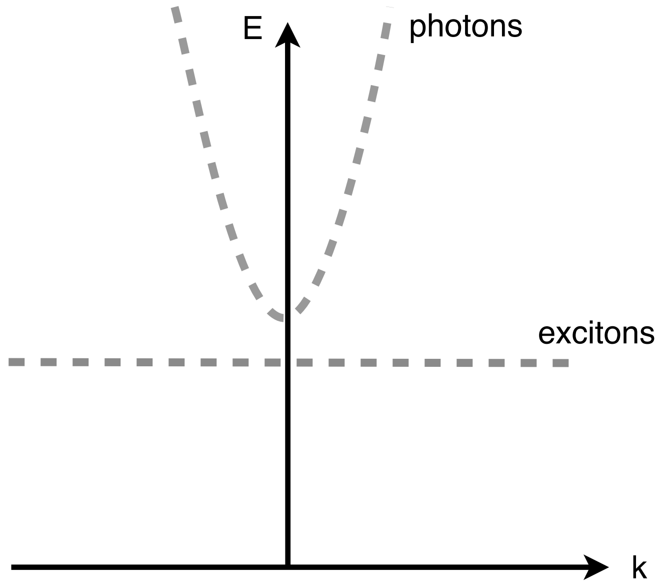
➡ spin degrees of freedom “fermionized”

spin-fermion mapping

$$\sigma_j^z = \hat{d}_j^\dagger \hat{d}_j - \hat{c}_j^\dagger \hat{c}_j ,$$
$$\sigma_j^+ = \hat{d}_j^\dagger \hat{c}_j , \sigma_j^- = \hat{c}_j^\dagger \hat{d}_j$$

- two independent fermion species, each obeying Dirac algebra
- two-species fermion bilinears obey the spin algebra

Microscopic Origin



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spin-fermion mapping

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$$\sigma_j^+ = \hat d_j^\dagger \hat c_j, \sigma_j^- = \hat c_j^\dagger \hat d_j$$

- interpretation:

σ_j^+ $|1\rangle$ $\hat d_j^\dagger \hat c_j$

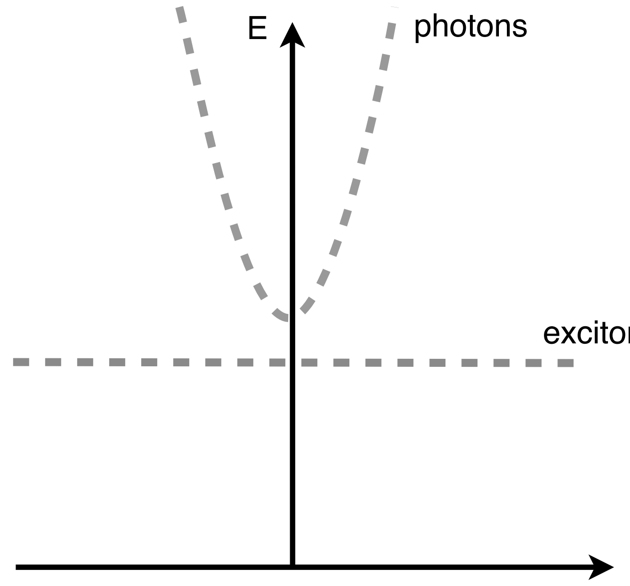
$|0\rangle$

$|\mathrm{vac}\rangle$

$\hat d_j^\dagger$ creates state $|1\rangle$ with exciton

$\hat c_j^\dagger$ creates state $|0\rangle$ w. ex. absent follows M. H. Szymańska, J. Keeling, P. B. Littlewood, arXiv:1206.1784

Microscopic Origin



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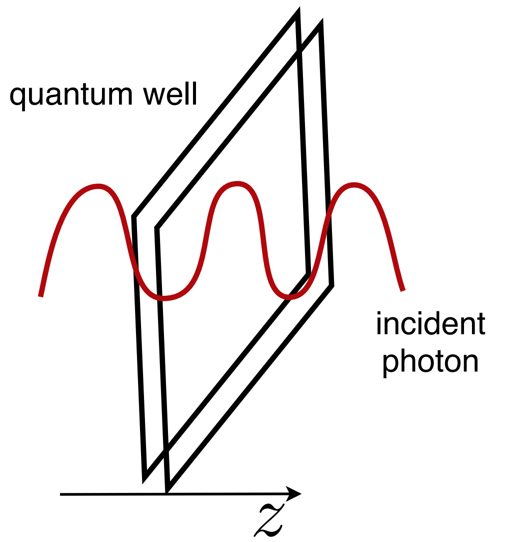
    ➡ spin degrees of freedom "fermionized"

  - photons: collection of plane waves with quadratic dispersion

$$H_{\text{ph}} = \sum_{\mathbf{p}} \hbar\omega_{\mathbf{p}} \hat{\Psi}_{\mathbf{p}}^\dagger \hat{\Psi}_{\mathbf{p}} \qquad \hbar\omega_{\mathbf{p}} = \hbar\omega_0 + \frac{\hbar^2 \mathbf{p}^2}{2m_{\text{ph}}}$$



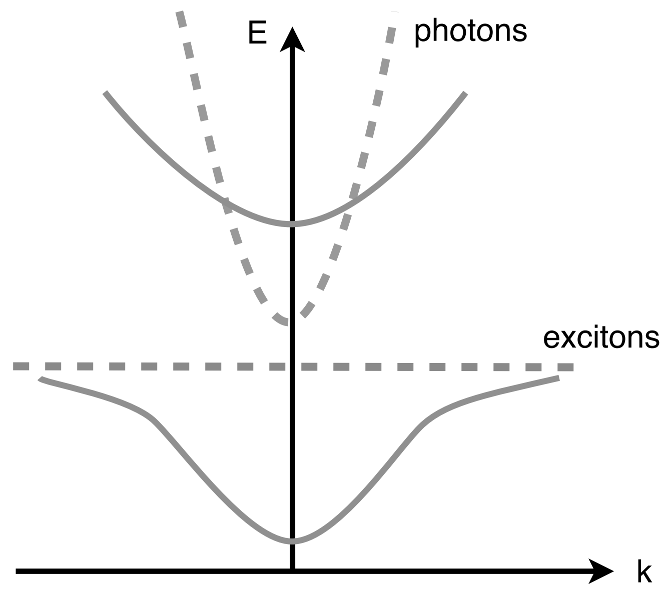
$$\hbar\omega_{\mathbf{p}}^{3\text{D}} = \hbar\omega_0' + \hbar\sqrt{p_x^2 + p_y^2 + m_{\text{ph}}^2}$$

$$\approx \hbar\omega_0 + \frac{\hbar^2 \mathbf{p}^2}{2m_{\text{ph}}}$$

2D in-plane momentum

follows M. H. Szymanska, J. Keeling, P. B. Littlewood, arXiv:1206.1784

Microscopic Origin



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    ➡ spin degrees of freedom "fermionized"

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  - hybridization: photon can create exciton coherently

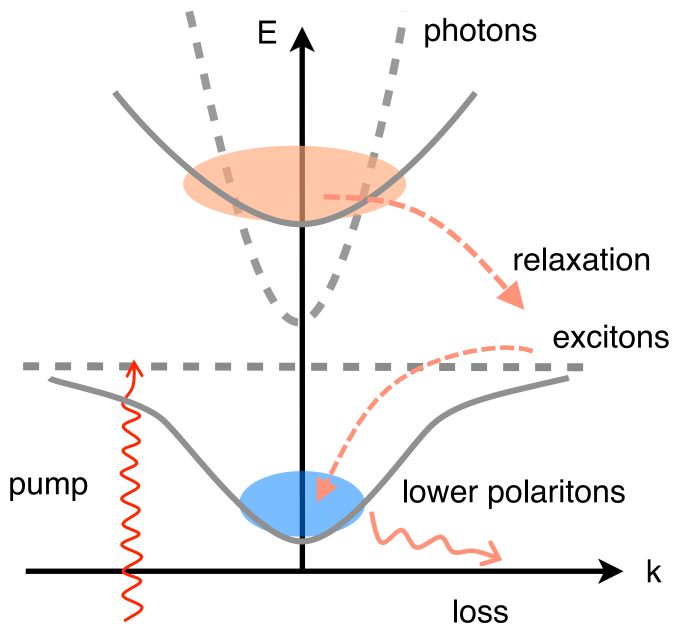
$$H_{\rm int} = \sum g_j \Psi_{\mathbf{p}}^\dagger \sigma_j^- = \sum g_j \Psi_{\mathbf{p}}^\dagger \hat{c}_j^\dagger \hat{d}_j$$

hybridization term

- interconversion of photons into excitons
- formally: cubic non-linearity
- model represents (Hamiltonian part of) a multimode laser model

follows M. H. Szymanska, J. Keeling, P. B. Littlewood, arXiv:1206.1784

Microscopic Origin



- include pump and dissipation: Keldysh formulation

$$S = \iint_{-\infty}^{\infty} dt dt' \left[\sum_j \Lambda_j^*(t) G_j^{-1}(t,t') \Lambda_j(t') + \sum_{\mathbf{p}} \Psi_{\mathbf{p}}^*(t) D_{(0),\mathbf{p}}^{-1}(t,t') \Psi_{\mathbf{p}}(t') \right] \qquad \Lambda_j = \begin{pmatrix} d_{c,j} \\ c_{c,j} \\ d_{q,j} \\ c_{q,j} \end{pmatrix}$$

- photon inverse Green's function:

$$D_{(0),\mathbf{p}}^{-1}(t,t') = \begin{pmatrix} 0 & i\hbar\partial_t - \hbar\omega_{\mathbf{p}} - i\kappa_c \\ i\hbar\partial_t - \hbar\omega_{\mathbf{p}} + i\kappa_c & 2i\kappa_c \end{pmatrix}$$

✓ photons decay: $\kappa_c > 0$

✓ bath assumed Markovian

- fermion (exciton) inv. Green's function and cubic non-linearity

$$G_j^{-1} = \begin{pmatrix} 0 & -\lambda_{\mathrm{q}}(t) & i\hbar\partial_t - \epsilon_j - i\gamma_x & -\lambda_{\mathrm{cl}}(t) \\ -\lambda_{\mathrm{q}}^*(t) & 0 & -\lambda_{\mathrm{cl}}^*(t) & i\hbar\partial_t + \epsilon_j - i\gamma_x \\ i\hbar\partial_t - \epsilon_j + i\gamma_x & -\lambda_{\mathrm{cl}}(t) & 2i\gamma_x F_D & -\lambda_{\mathrm{q}}(t) \\ -\lambda_{\mathrm{cl}}^*(t) & i\hbar\partial_t + \epsilon_j + i\gamma_x & -\lambda_{\mathrm{q}}^*(t) & 2i\gamma_x F_C \end{pmatrix}$$

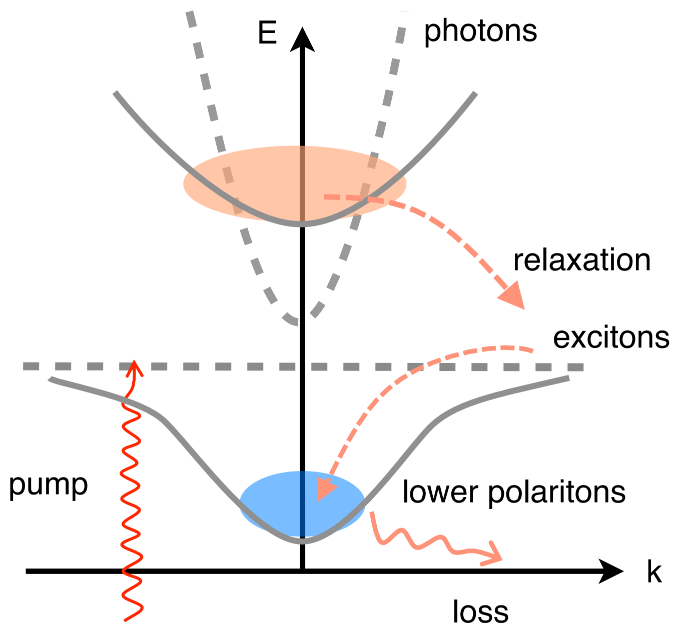
$$\lambda_{\mathrm{q}}(t) = \sum_{\mathbf{p}} g_j \Psi_{\mathbf{p},q}(t)/\sqrt{2} \qquad \lambda_{\mathrm{cl}}(t) = \sum_{\mathbf{p}} g_j \Psi_{\mathbf{p},cl}(t)/\sqrt{2}$$

✓ excitons are pumped

✓ "fermion distribution functions" F_D, F_C describe exciton inversion (cf. laser)

$$\boxed{N_0 = -(F_D - F_C)/2}$$

Microscopic Origin



- effective polariton action after fermion (Gaussian) integration:

$$S = \iint_{-\infty}^{\infty} dt dt' \sum_{\mathbf{p}} \Psi^*_{\mathbf{p}}(t) D^{-1}_{(0),\mathbf{p}}(t,t') \Psi_{\mathbf{p}}(t')$$

$$-i \sum_j \mathrm{Tr} \left\{ \ln G_j^{-1} \right\} [\Psi^*_{\mathbf{p}}, \Psi_{\mathbf{p}}]$$

➡ due to cubic non-linearity: fermion fluctuation term is a function of the photon field

➡ Landau-Ginzburg theory: expand to quartic order

- here we proceed on the level of the equation of motion
- further, we study homogeneous field configurations (mean fields)

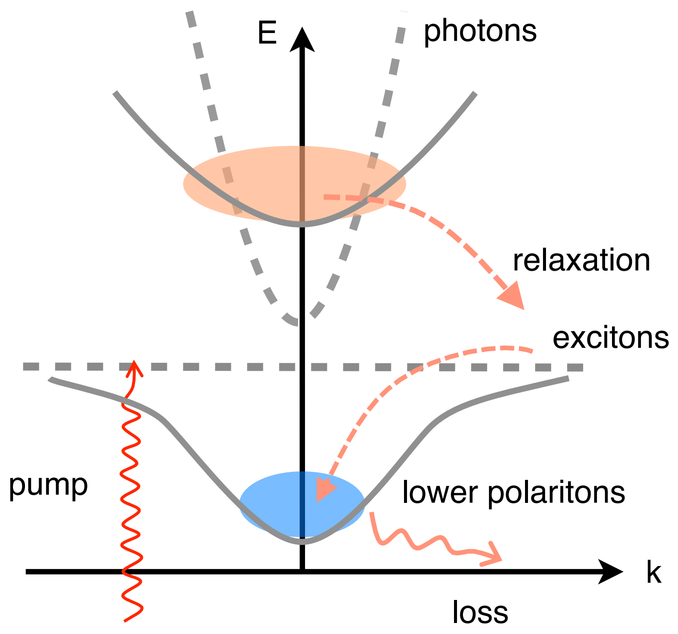
condensation in zero momentum mode (BEC)

polariton condensate amplitude

phase adjusting stat. state

$$\Psi_{c,\mathbf{p}} = \phi_0 \sqrt{2} \delta(\mathbf{p}) e^{-i\mu_S t}$$

Microscopic Origin



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  - ➡ Landau-Ginzburg theory: expand to quartic order

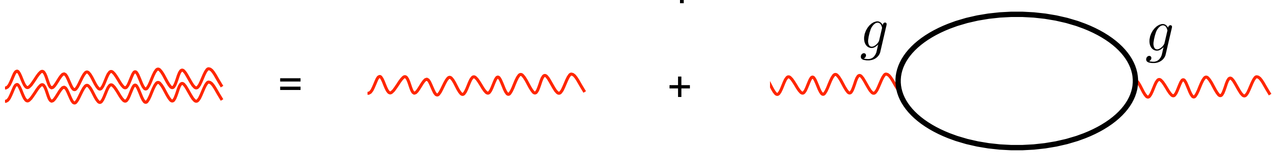
- here we proceed on the level of the equation of motion
- further, we study homogeneous field configurations (mean fields)

$$\Psi_{q,\mathbf{q}} = 0$$

$$\Psi_{c,\mathbf{p}} = \phi_0 \sqrt{2} \delta(\mathbf{p}) e^{-i\mu_S t}$$

$$0 \stackrel{!}{=} \frac{\delta S}{\delta \Psi_{q,\mathbf{p}}}\Big|_{\Psi_{c,\mathbf{p}}=\phi_0} = \Big(\underbrace{\hbar\omega_0 - \mu_S - i\kappa_c}_{\text{from bare photon inverse Green's function}} \underbrace{- I(\phi_0^*\phi_0)}_{\text{renormalization from fermion (exciton) fluctuations generates non-linearity}} \Big) \phi_0$$

- interpretation: correction due to interconversion processes



follows M. H. Szymanska, J. Keeling, P. B. Littlewood, arXiv:1206.1784

Microscopic Origin

- homogenous polariton equation of motion

$$0 \stackrel{!}{=} \frac{\delta S}{\delta \Psi_{q,\mathbf{p}}}\Big|_{\Psi_{c,\mathbf{p}}=\phi_0} = \Big(\underbrace{\hbar\omega_0 - \mu_S - i\kappa_c}_{\text{from bare photon inverse Green's function}} \underbrace{- I(\phi_0^*\phi_0)}_{\text{renormalization from fermion (exciton) fluctuations generates non-linearity}}\Big)\phi_0$$

$$I(\phi_0^*\phi_0) = \frac{N_0}{2}\sum_j g_j^2 \frac{-\epsilon_j + \mu_S/2 + i\gamma_x}{E_j^2 + \gamma_x^2} \qquad E_j^2 = (\epsilon_j - \mu_S/2)^2 + g_j^2\phi_0^*\phi_0$$

$$= N_0(a_1 + ia_2 + (b_1 + ib_2)\phi^*\phi + \dots)$$

- in particular: signs of the dissipative coefficients $a_2 > 0, b_2 < 0$

  ➡ in the case of population inversion $N_0 < 0$ exciton fluctuation correction acts as pump

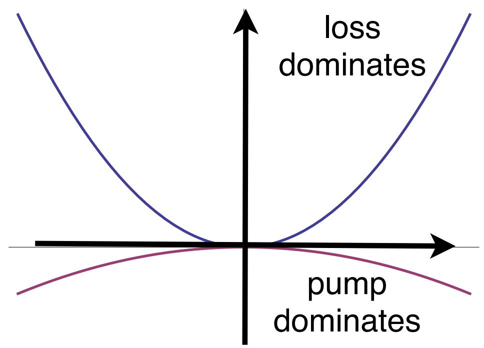
  ➡ effective pump exceeds loss: polariton condensation instability

  ➡ condensation threshold for homogeneous couplings and exciton energies

  $$g_j = g, \epsilon_j = \epsilon$$
  $$j = 1, \dots, n$$

  total inversion = $nN_0 = 2\kappa_c\gamma_x/g^2$

  ➡ fully analogous to a laser threshold



Polariton Condensation and Spontaneous Symmetry Breaking

- generalize homogenous polariton equation of motion to inhomogeneous one

$$\frac{\delta S}{\delta \Psi_{q,\mathbf{x}}} \stackrel{!}{=} 0 \Leftrightarrow \qquad i\partial_t \phi_c = \Big[\cancel{-\frac{\nabla^2}{2m}} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\,|\phi_c|^2 \Big]\,\phi_c + \cancel{\phi_q}$$

propagation

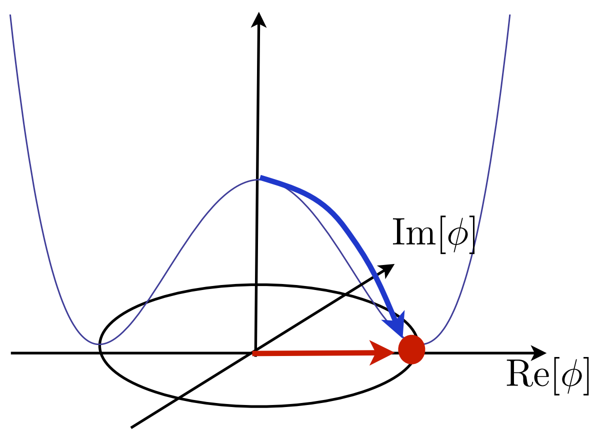
pump & loss rates

elastic collisions

two-body loss

  - valid for slow/long wavelength modes
  - we write the noise field (omitted before)

- Condensation: overdamped motion in Mexican hat potential



condensate density

- for dominant pump: $\gamma_p > \gamma_l \quad \Rightarrow |\phi_0|^2 = \frac{\gamma_p - \gamma_l}{\kappa}$

  "chemical potential" $\Rightarrow \mu = \lambda |\phi_0|^2$

- an instance of spontaneous symmetry breaking:
  - Equation of motion/action has symmetry of global phase rotations

$$\phi_0 \to \phi_0' = e^{\mathrm{i}\alpha}\phi_0$$

  - Symmetry broken by stationary condensed state with definite phase

Symmetry breaking and Goldstone Theorem

- Goal: understand the nature of the low momentum modes and comparison to equilibrium
- First key step: Goldstone theorem

- Obtain action from equation of motion by integration wrt. the noise field:

$$S = \int_{t,\mathbf{x}} \Big\{ \tfrac{1}{2}(\phi_c^*(t,\mathbf{x}), \phi_q^*(t,\mathbf{x})) \begin{pmatrix} 0 & P^A \\ P^R & i(\gamma_l + \gamma_p) \end{pmatrix} \begin{pmatrix} \phi_c(t,\mathbf{x}) \\ \phi_q(t,\mathbf{x}) \end{pmatrix}$$

$$-\tfrac{1}{2}[(\lambda - \mathrm{i}\kappa)|\phi_c(t,\mathbf{x})|^2 \phi_c(t,\mathbf{x})\phi_q^*(t,\mathbf{x}) + c.c.] \Big\}$$

$$P^R = \mathrm{i}\partial_t - (-\frac{\nabla^2}{2m_{\rm ph}} + \mu - \mathrm{i}(\gamma_l - \gamma_p)/2)$$

$$P^A = (P^R)^\dagger$$

- this action manifestly has the symmetry / invariance under global phase rotations (U(1) symmetry)

$$\phi_c \to \phi_c' = e^{\mathrm{i}\alpha}\phi_c$$

$$\phi_q \to \phi_q' = e^{\mathrm{i}\alpha}\phi_q$$

for the same rotation angle

$$\alpha$$

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$$-\tfrac{1}{2}[(\lambda - \mathrm{i}\kappa)|\phi_c(t,\mathbf{x})|^2 \phi_c(t,\mathbf{x})\phi_q^*(t,\mathbf{x}) + c.c.]\Big\} \qquad P^A = (P^R)^\dagger$$

- Introduce the effective action as the “action plus all fluctuations”

$$\underbrace{e^{i\Gamma[\phi^*_{c,q},\phi_{c,q}]}}_{\text{field expectation value, “classical field”}} = \int \underbrace{\mathcal{D}(\delta\varphi^*_{c,q}, \delta\varphi_{c,q})}_{\text{sum over all possible fluctuation configurations}} e^{iS[\phi^*_{c,q} + \underbrace{\delta\varphi^*_{c,q}}_{\text{fluctuation around “classical field”}}, \phi_{c,q} + \delta\varphi_{c,q}]}$$

- NB: this obtains formally by Legendre transformation of $\log Z[j^*_{c,q}, j_{c,q}]$
- field equation: generalization of action principle $\dfrac{\delta\Gamma}{\delta\phi_\nu(t,\mathbf{x})} = \dfrac{\delta\Gamma}{\delta\phi^*_\nu(t,\mathbf{x})} = 0$

Symmetry breaking and Goldstone Theorem

- simple proof of Goldstone theorem using the effective action

$$e^{i\Gamma[\phi^*_{c,q},\phi_{c,q}]} = \int \mathcal{D}(\delta\varphi^*_{c,q},\delta\varphi_{c,q}) e^{iS[\phi^*_{c,q}+\delta\varphi^*_{c,q},\phi_{c,q}+\delta\varphi_{c,q}]}$$

- Goal: Assume symmetry is broken => there exists a gapless mode (zero excitation energy cost/ zero damping at zero momentum)

$$\omega(\mathbf{q}=0)=0 \qquad \text{i.e. study} \qquad \omega = \mathbf{q} = 0$$

- decompose: $\Gamma[\phi^*_\nu,\phi_\nu] = \Gamma_h[\phi^*_\nu,\phi_\nu] + \Gamma_n[\partial_\tau\phi^*_\nu,\partial_{x_i}\phi_\nu,\partial_{x_i}\phi^*_\nu,\partial_{x_i}\phi_\nu,...]$ $\qquad \nu = c,q$

homogeneous: zero freq. / mom. sector — non-homogeneous

➡ sufficient to analyze Γ_h

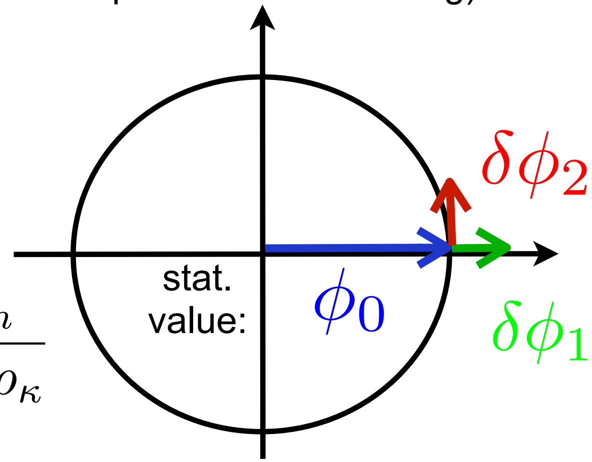
- U(1) invariance $\Rightarrow \Gamma_h[\phi^*_\nu,\phi_\nu] = \Gamma_h[\rho_\mu]$ $\qquad \rho_\mu = \phi^*_\nu\phi_{\nu'}$ all U(1) invariant combinations, but nothing else!

$\mu = \{cc; cq; qc; qq\}$

Symmetry breaking and Goldstone Theorem

$$\Gamma_h[\phi^*_\nu, \phi_\nu] = \Gamma_h[\rho_\mu] \qquad \rho_\mu = \phi^*_\nu \phi_{\nu'}$$

choice of field coordinates (due to spontaneous SB: wlog)



- assume SSB $\quad \phi_c \equiv \phi_0 \neq 0 \quad$ but $\quad \phi_q = 0$

- properties of excitation spectrum: R/A sectors of second derivative, gap/mass matrix:

$$M_{ij} \equiv \frac{\partial^2 \Gamma_h}{\partial\chi_i \partial\chi_j}\Big|_{\text{stat}} = \sum_\mu \underbrace{\frac{\partial^2 \rho_\mu}{\partial\chi_i\partial\chi_j}\frac{\partial\Gamma_h}{\partial\rho_\mu}}_{=0 \text{ in R/A sectors}} + \sum_{\mu,\kappa} \frac{\partial\rho_\mu}{\partial\chi_i}\frac{\partial\rho_\kappa}{\partial\chi_i}\frac{\partial^2\Gamma_h}{\partial\rho_\mu\partial\rho_\kappa}$$

- key implication of broken symmetry: first term vanishes in R/A sectors due to homogenous “equation of motion”

$$\frac{\partial\Gamma_h}{\partial\chi_i} = \sum_\mu \frac{\partial\rho_\mu}{\partial\chi_i}\frac{\partial\Gamma_h}{\partial\rho_\mu} \overset{!}{=} 0 \quad \forall i$$

➡ excitation matrix must be of the form (exercise)

$$\boxed{M^R_{ij} = 2\rho_0^2 \begin{pmatrix} \lambda & 0 \\ \mathrm{i}\kappa & 0 \end{pmatrix}}$$

λ, κ real: second derivatives of Γ_h

$\rho_0 = \phi_0^2$

➡ U(1) invariance of full theory implies existence of gapless mode (zero eigenvalue of mass matrix)

Nature of Low Momentum Dynamics

- Summary: Goldstone theorem

Consider a theory which is invariant under a continuous global symmetry transformation. Assume the symmetry is broken spontaneously. Then, there are gapless modes (Goldstone modes).

➡ NB: no reference to equilibrium or non-equilibrium nature

➡ but to symmetry and a qualitative property of the state (SSB)

➡ no information on the form of the low momentum modes

- now, construct the excitations

  - most general form of excitation matrix in SSB phase

$$P^R(\omega,\mathbf{q}) = \delta P^R(\omega,\mathbf{q}) - M^R \qquad \text{with} \quad \delta P^R(\omega=\mathbf{q}=0)=0$$

$$M^R_{ij} = 2\rho_0^2 \begin{pmatrix} \lambda & 0 \\ \mathrm{i}\kappa & 0 \end{pmatrix} \qquad \delta P^R(\omega,\mathbf{q}^2) = i\hat{Z}\omega - \hat{A}\mathbf{q}^2 \qquad \text{with} \quad \hat{Z},\hat{A} \quad \text{real 2x2 matrices}$$

  - for the above polariton action, we have explicitly

$$\hat{Z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{A} = \frac{1}{2m_{\mathrm{ph}}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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  - calculate excitation spectrum from poles of Green's function or $\det P^R(\omega, \mathbf{q}) \stackrel{!}{=} 0$
  - but no matter how complicated, we always have diffusive behavior

$$\omega(\mathbf{q}) = -\mathrm{i} D_{\mathrm{eff}}\, \mathbf{q}^2 \ \text{ for } \mathbf{q} \to 0$$

$$D_{\mathrm{eff}} > 0 \ \text{ for } \kappa > 0$$

$$D_{\mathrm{eff}} = \frac{\lambda}{\;} \qquad \text{in example above}$$

Comparison to thermodynamic equilibrium

- in the non-equilibrium situation, we found based on U(1) symmetry:

$$\omega(\mathbf{q}) = -\mathrm{i}D_{\text{eff}}\,\mathbf{q}^2 \ \text{for}\ \mathbf{q} \to 0$$

diffusive Goldstone mode

- in equilibrium symmetry broken phase (BEC), it is well known

$$\omega(\mathbf{q}) = c|\mathbf{q}| \ \text{for}\ \mathbf{q} \to 0$$

propagating Goldstone (sound) mode

➡ the difference is traced back to the absence of exact particle number conservation out of equilibrium

- here: open system, incoherent particle loss and gain
- equilibrium: closed system, particle number conserved
- formally: additional U(1) symmetry in closed system

  - indeed, two symmetry generators on the contour:

$$\begin{pmatrix} \varphi'_+(t,\mathbf{x}) \\ \varphi'_-(t,\mathbf{x}) \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\alpha_+} & 0 \\ 0 & e^{\mathrm{i}\alpha_-} \end{pmatrix} \begin{pmatrix} \varphi_+(t,\mathbf{x}) \\ \varphi_-(t,\mathbf{x}) \end{pmatrix}$$

  - we focused above on

$$\alpha_+ = \alpha_-$$

i.e.

$$\alpha_c = (\alpha_+ + \alpha_-)/2 \neq 0,$$

$$\alpha_q = (\alpha_+ - \alpha_-)/2 = 0$$

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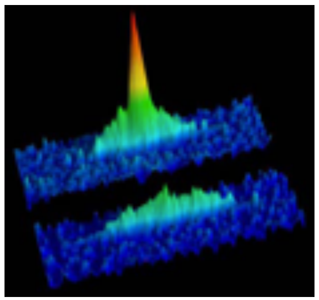
  - closed system: additional invariance under α_q
  - indeed: Noether charge for α_q is the particle number
  - implication for mass matrix:

$$M^R_{ij} = 2\rho_0^2 \begin{pmatrix} \lambda & 0 \\ \cancel{\mathrm{i}\kappa} & 0 \end{pmatrix}$$

purely real; plus further constraints on $\hat{Z}, \hat{A}$

  - consequence: dominant hydrodynamic sound mode

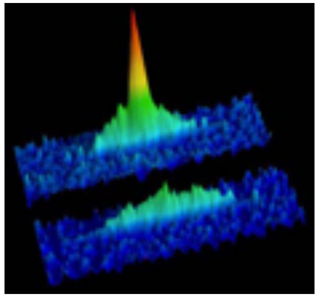
Critical Phenomena and Universality
(Equilibrium)



$$\simeq$$

Critical Phenomena and Universality (Equilibrium)

- Universality: The art of systematically forgetting about details



Bose-Einstein Condensate

$$\simeq$$

planar magnets

at the critical point

$$\tau = \frac{T - T_c}{T} \to 0$$

- The experimental witnesses: Critical exponents, e.g.

$$\langle \phi^*(r)\phi(0)\rangle \sim \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

correlation length

$$\xi \sim |\tau|^{-\nu} \to \infty$$

- The exponents:

ν "mass/gap exponent"

η "anomalous dimension"

nontrivial statement:

no more independent exponents *

than these!

Critical Phenomena and Universality (Equilibrium)

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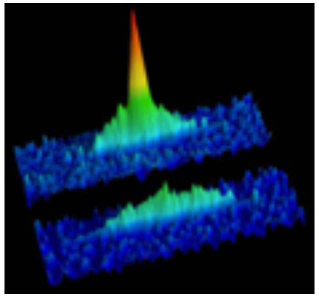

Bose-Einstein Condensate

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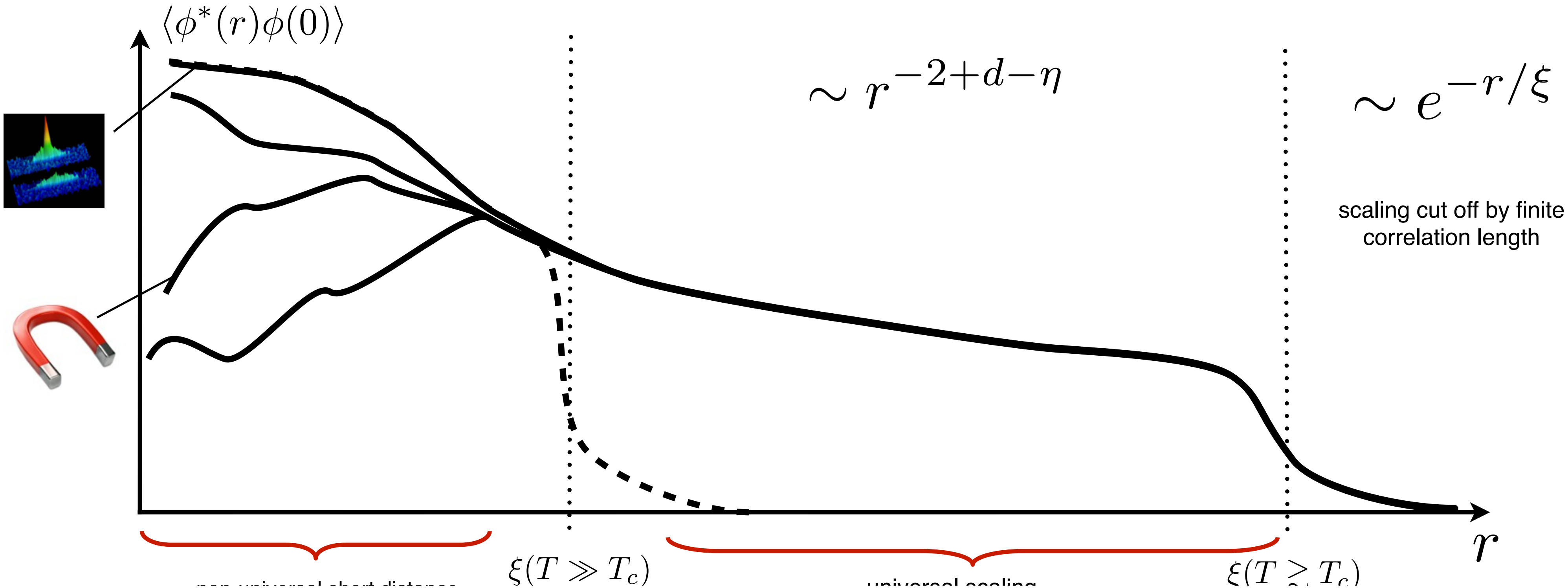
planar magnets

- The physical picture: universality induced by divergent correlation length



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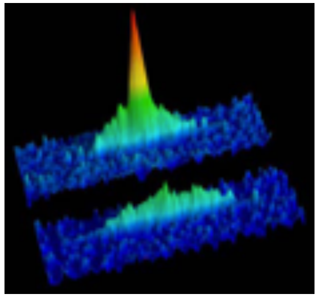

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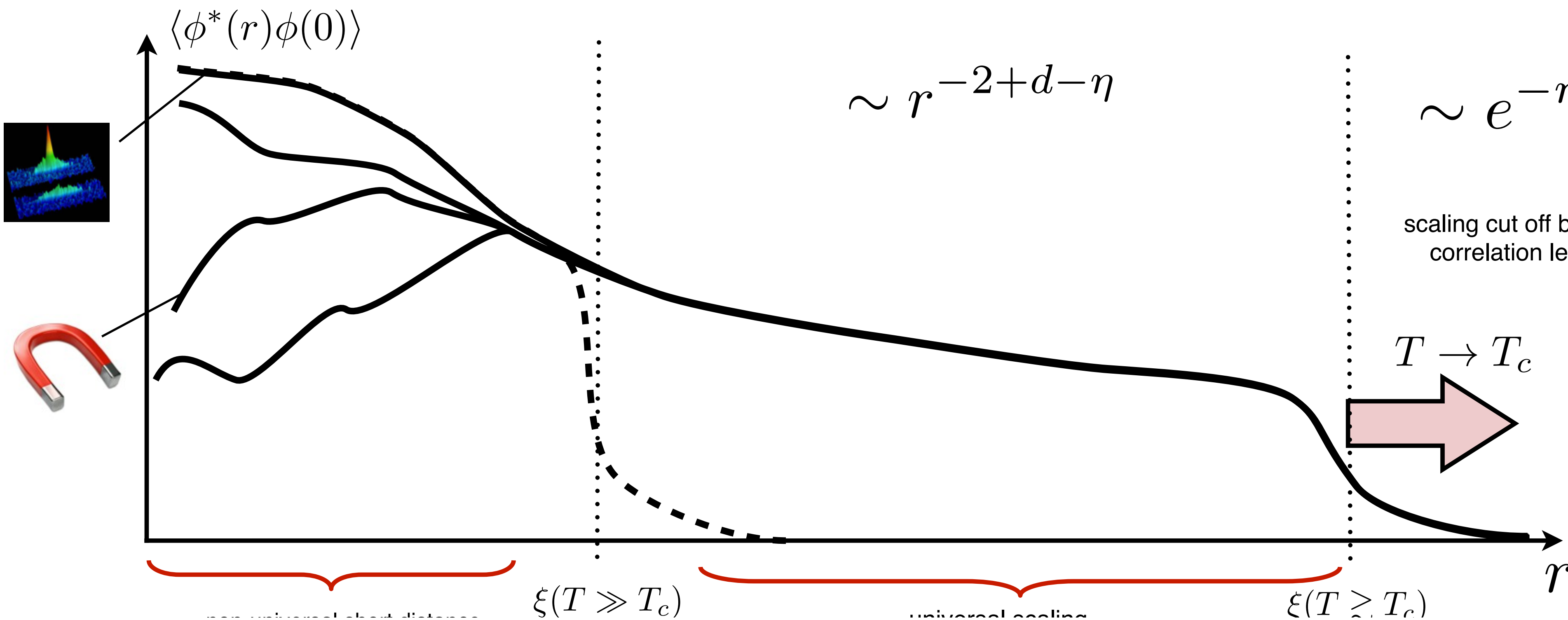

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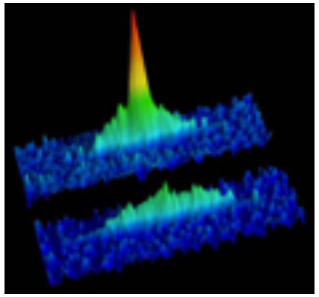


Bose-Einstein Condensate

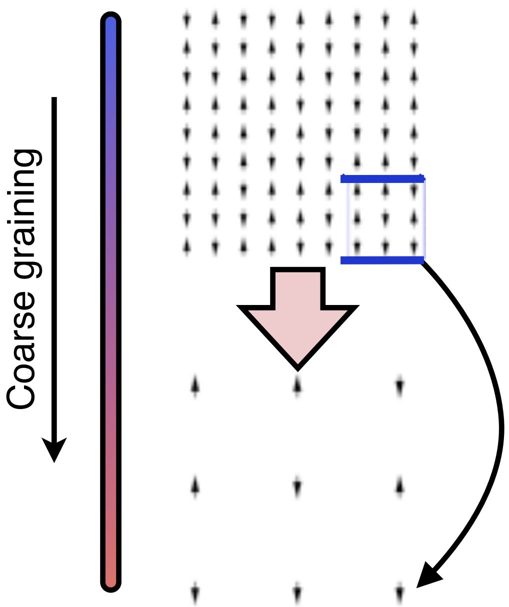
$\simeq$

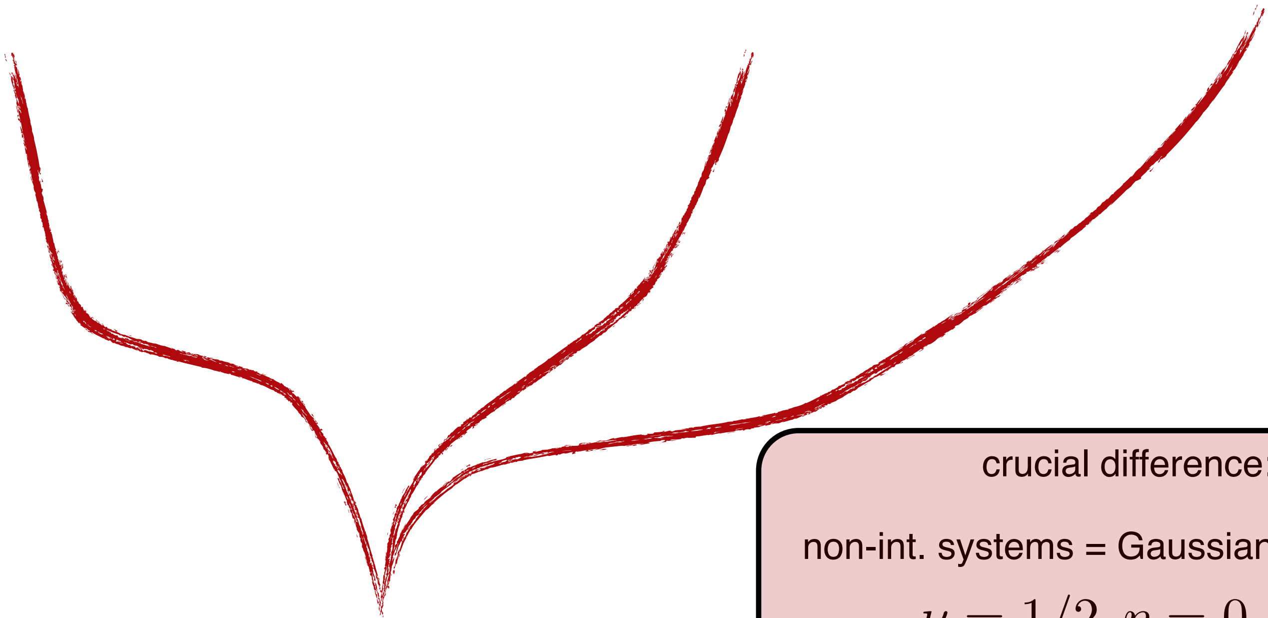
planar magnets

- The description: Renormalization group

other systems...

UV: microscopic physics





IR: long-wavelength

Wilson-Fisher fixed point

crucial difference:

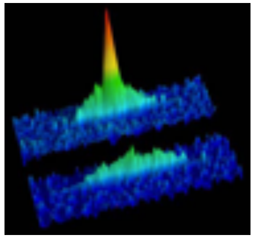
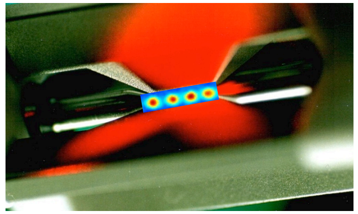
non-int. systems = Gaussian fixed point

$$\nu = 1/2, \eta = 0$$

interacting systems = WF fixed point

Universality Classes (Equilibrium)

- Universality classes: Memory of symmetries is kept

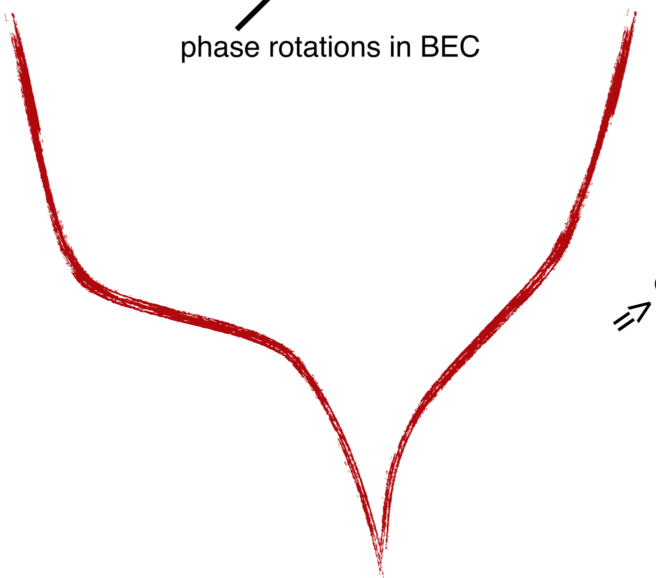
 $\simeq$  $\neq$  $\simeq$ 

Bose-Einstein Condensate | planar magnets | trapped ions | liquid-gas transition i carbon-dioxide

- Symmetries: $U(1) \simeq O(2)$ Z_2

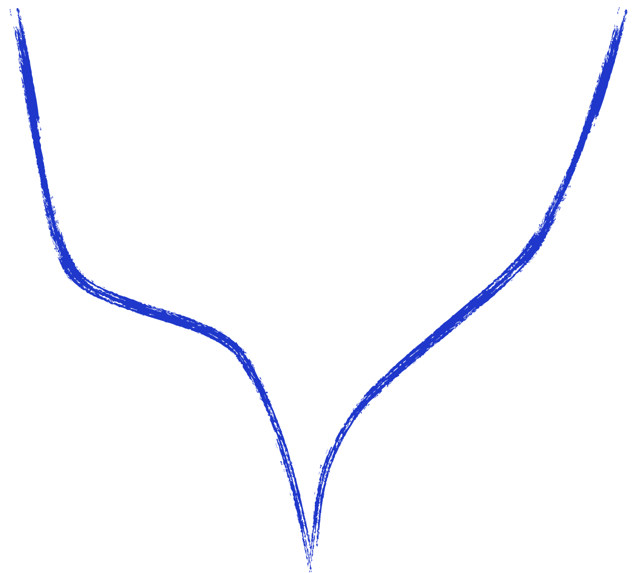
phase rotations in BEC

Coarse graining



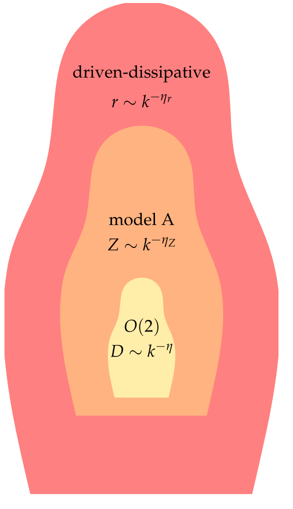
~ 80 stable elements
=> O(10^10) possible compounds
~ 10^23 particles

but only a handful
universality classes



"O(2) universality class" "Ising universality class"

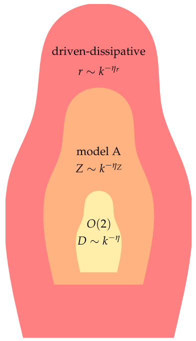
Criticality in Driven-Dissipative Many-Body Systems



L. Sieberer, S. Huber, E. Altman, SD, PRL 2013;
in preparation

Criticality in Driven-Dissipative Many-Body Systems

- Questions and challenges:
  - Physics: Understanding the nature of driven-dissipative phase transitions
    - Universality class: Can non-equilibrium conditions modify equilibrium criticality, given massive loss of memory?
    - Thermalization of driven-dissipative systems?
    - Decoherence?

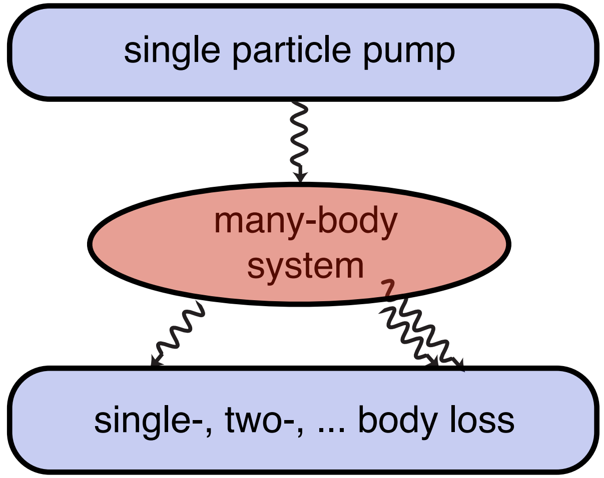


  - Methods:
    - Construct efficient quantum field theoretical framework for out-of-equilibrium criticality

$$e^{i\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi e^{iS_M[\Phi+\delta\Phi]}$$

Microscopic model: Many-Body Quantum Master Equation

- universal microscopic model: many-body master equation



$$\partial_t \rho = -i[H, \rho] + \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu\right) \hat{\phi}_{\mathbf{x}} + \frac{\lambda}{2} (\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}})^2$$

$$\mathcal{L}[\rho] = \underbrace{\gamma_p \int_{\mathbf{x}} [\hat{\phi}^\dagger_{\mathbf{x}} \, \rho \, \hat{\phi}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}}, \rho\}]}_{\text{single particle pump}} \quad + \quad \underbrace{\gamma_l \int_{\mathbf{x}} [\hat{\phi}_{\mathbf{x}} \, \rho \, \hat{\phi}^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}}, \rho\}]}_{\text{single particle loss}} \quad +$$

$$\underbrace{\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}} \, \rho \, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^2_{\mathbf{x}}, \rho\}]}_{\text{two particle loss}}$$

cf. Quantum Optics:

- single mode, H=0, semiclassical approximation: effective laser threshold equations

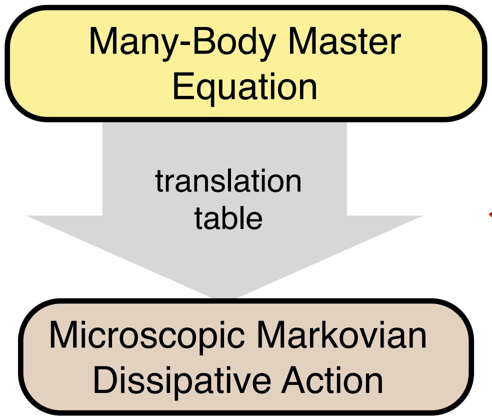
cf. Many-Body Physics:

- continuum of spatial degrees of freedom: infrared divergence

➡ second quantized operator formalism inappropriate

➡ need method transfer: develop efficient functional many-body techniques

The Theoretical Approach



- Step 1: translation table

many-body master equation

$$\partial_t \rho = -i[H, \rho] + \mathcal{L}[\rho]$$

$$\Updownarrow$$

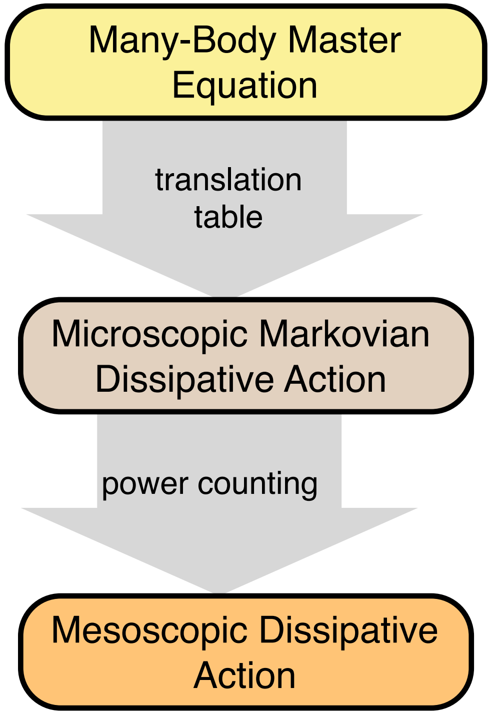
$$\mathrm{e}^{\mathrm{i}\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi \mathrm{e}^{\mathrm{i}S_M[\Phi+\delta\Phi]}$$

Markovian dissipative action

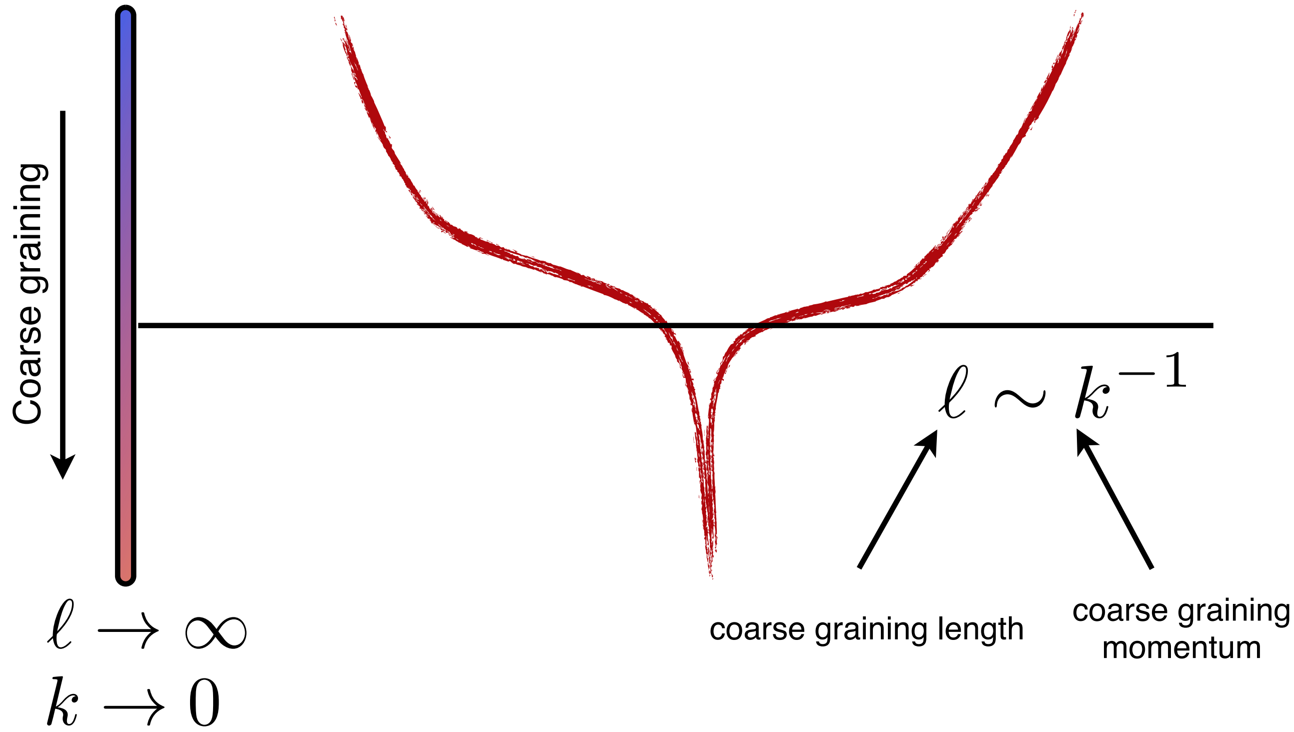
Keldysh real time functional integral

➡ Opens up the powerful toolbox of quantum field theory to driven-dissipative systems

The Theoretical Approach

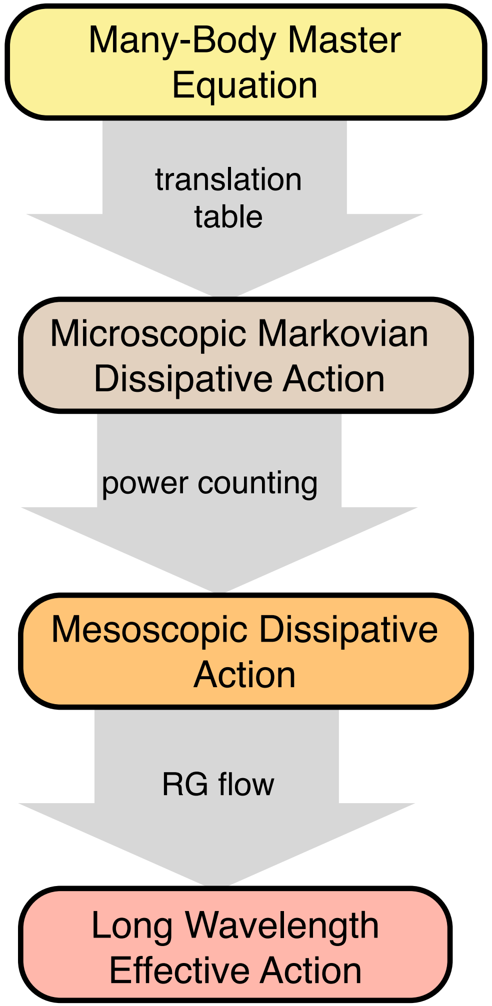


- Step 2: Canonical power counting: Classification of relevance of interactions at criticality



➡ Microscopic quantum model reduces exactly to phenomenological, classical stochastic model

The Theoretical Approach



- Step 3: Run functional renormalization group flow

Keldysh real time functional integral

$$e^{i\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi e^{iS_M[\Phi+\delta\Phi]}$$

$$\Updownarrow$$

$$\partial_k \Gamma_k = \frac{i}{2} \mathrm{Tr} \left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_k R_k \right]$$

Functional Renormalization Group equation

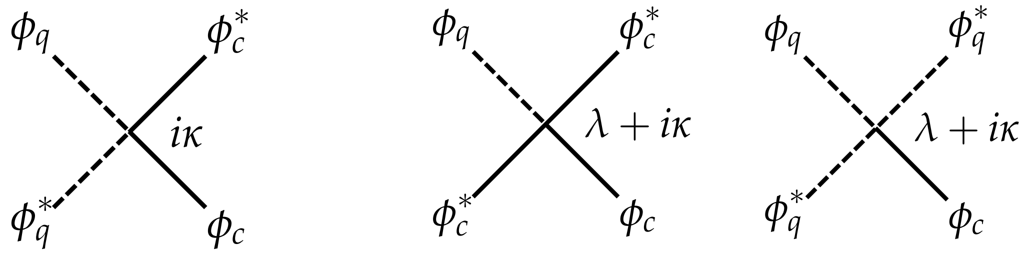
Wetterich, Z. Phys. 93
Keldysh closed syst.:
Gasenzer&, Phys. Lett. 08
Berges&, Nucl. Phys. B 09

➡ Discussion of the key phenomena:
- ➡ Decoherence
- ➡ Thermalization
- ➡ Universality

Microscopic markovian dissipative action

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \phi_c^* \phi_c \phi_q^* \phi_q - \frac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \phi_q^{*2} \phi_c \phi_q \right) + c.c. \right] \right\}$$

- Gaussian sector: inverse Green's function



- retarded/advanced $P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 - \mu + i(\gamma_l - \gamma_p)/2$

- Keldysh component $P^K = i(\gamma_l + \gamma_p)$

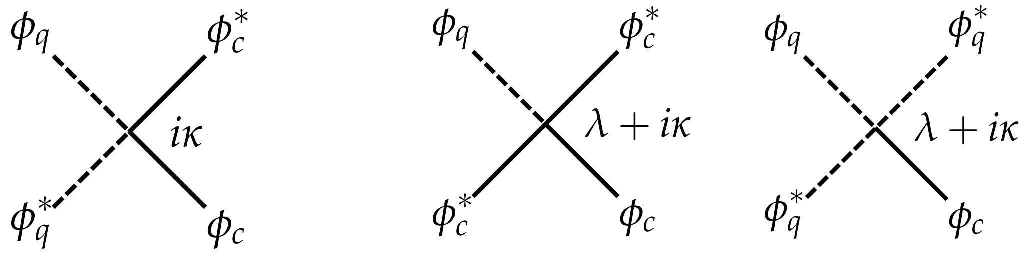
- Relation to single-particle observables:

$$-i\langle \phi_\sigma^* \phi_{\sigma'} \rangle = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} = \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix}^{-1}$$

Structuring the problem by power counting

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \phi_c^* \phi_c \phi_q^* \phi_q - \tfrac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \phi_q^{*2} \phi_c \phi_q \right) + c.c. \right] \right\}$$

- Gaussian sector at criticality:



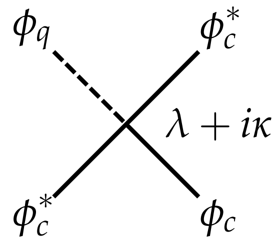
  - retarded/advanced $P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 \overbrace{- \mu + i\left(\gamma_l - \gamma_p\right)/2}^{\to 0} \sim q^2$
  - Keldysh component $P^K = i\left(\gamma_l + \gamma_p\right) \sim q^0$

- Canonical field dimensions: $[\phi_c] = \dfrac{d-2}{2} < [\phi_q] = \dfrac{d+2}{2}$

  - action is dimensionless: phase e^{iS} in the functional integral
  - quadratic/Gaussian sector: scaling dimensions of inverse Green's function known
  - intuitive: high order local couplings not relevant at large distances

Structuring the problem by power counting

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + \cancel{2i\kappa\phi_c^*\phi_c\phi_q^*\phi_q} - \tfrac{1}{2}\left[(\lambda + i\kappa)\left(\phi_c^{*2}\phi_c\phi_q + \cancel{\phi_q^{*2}\phi_c\phi_q}\right) + c.c. \right] \right\}$$



- Gaussian sector at criticality:

  - retarded/advanced $\quad P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 \overbrace{- \mu + i\left(\gamma_l - \gamma_p\right)/2}^{\to 0} \sim q^2$

  - Keldysh component $\quad P^K = i\left(\gamma_l + \gamma_p\right) \sim q^0$

- Canonical field dimensions: $\quad [\phi_c] = \dfrac{d-2}{2} < [\phi_q] = \dfrac{d+2}{2}$

➡ Local vertices with more than two quantum fields are irrelevant in the RG sense in d > 2

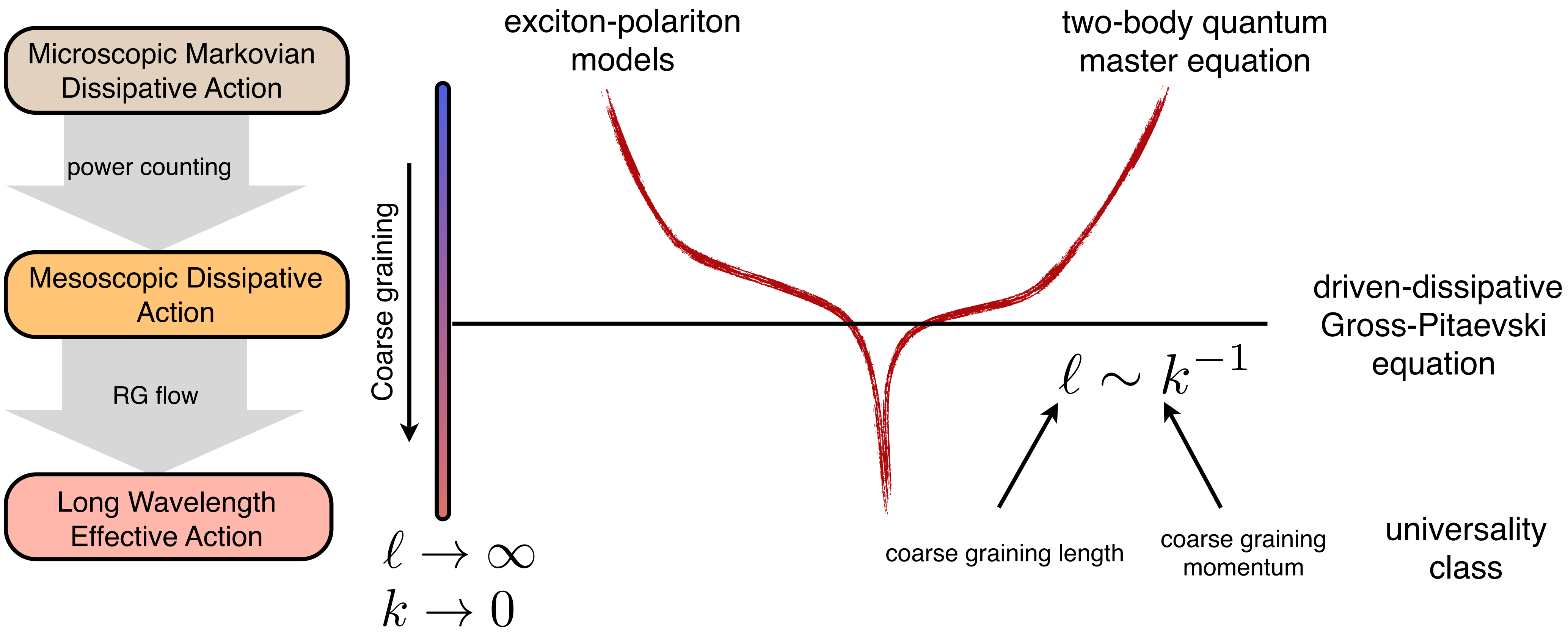
➡ massive diagrammatic simplification

➡ identical to phenomenological models of exciton-polariton condensates (Wouters and Carusotto PRL 06; Szymanska, Keeling, Littlewood PRL 04)

➡ Original quantum problem becomes a classical stochastic field theory

Power counting and exciton-polariton model

- example of “weak” universality



➡ many microscopic models collapse to an effective low energy model

➡ form dictated by microscopic symmetries

➡ universality class to be determined by calculation

Power Counting and "Classicality"

- physical interpretation: reduction to classical problem in d > 2

$$F \sim \frac{1}{\omega}$$

distribution function

F defined via

$$G^K = G^R F - F G^A$$

fluctuation-dissipation relation

- infrared mode occupation enhanced
- same scaling as in thermal equilibrium: $F_{\rm eq} = \frac{2T}{\omega}$
- equilibrium fluctuation-dissipation theorem

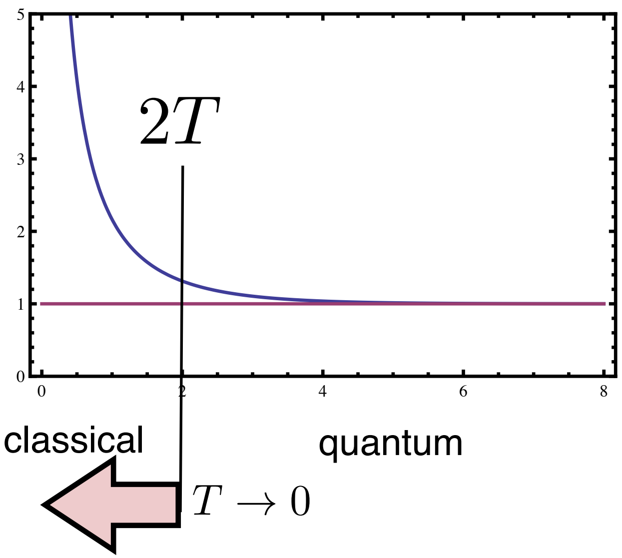
$$F_{\rm eq} = \coth \tfrac{\omega}{2T} = 2n_{(\frac{\omega}{T})} + 1$$

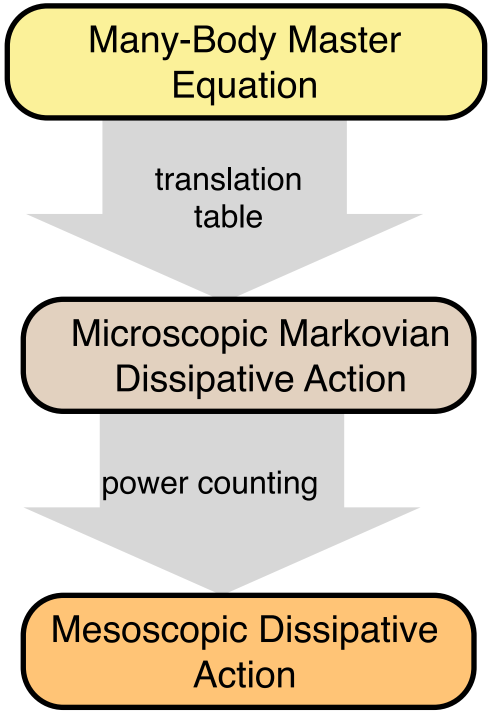
$$= \mathrm{sgn}(\omega), \quad T = 0$$

➡ no states but ground state occupied

$$= \frac{2T}{\omega}, \quad \omega \ll T$$

➡ states with low energies highly occupied





Power Counting and “Classicality”

- physical interpretation: reduction to classical problem in d > 2

$$F \sim \frac{1}{\omega}$$

distribution function

$$G^K = G^R F - F G^A$$

fluctuation-dissipation relation

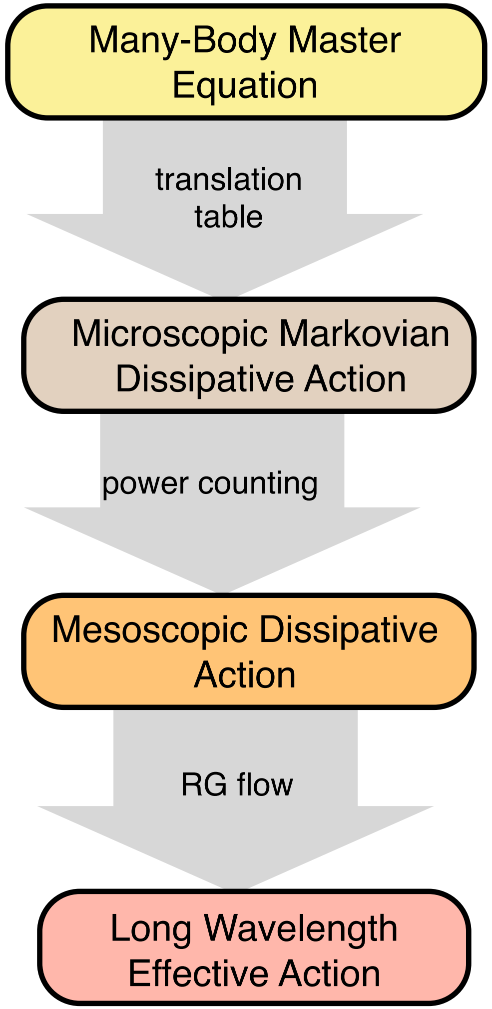
- infrared mode occupation enhanced
- same scaling as in thermal equilibrium $\quad F_{\mathrm{eq}} = \dfrac{2T}{\omega}$
- similar findings: Mitra et al., PRL 2006 (Ising model); Mitra and Rosch, PRL 2010 (Kondo model)

- key differences to equilibrium relaxational models

Halperin and Hohenberg, RMP 76

➡ arbitrary complex coupling parameters, independent coherent and dissipative dynamics: driven system at mesoscopic scale

➡ thermal equilibrium not enforced



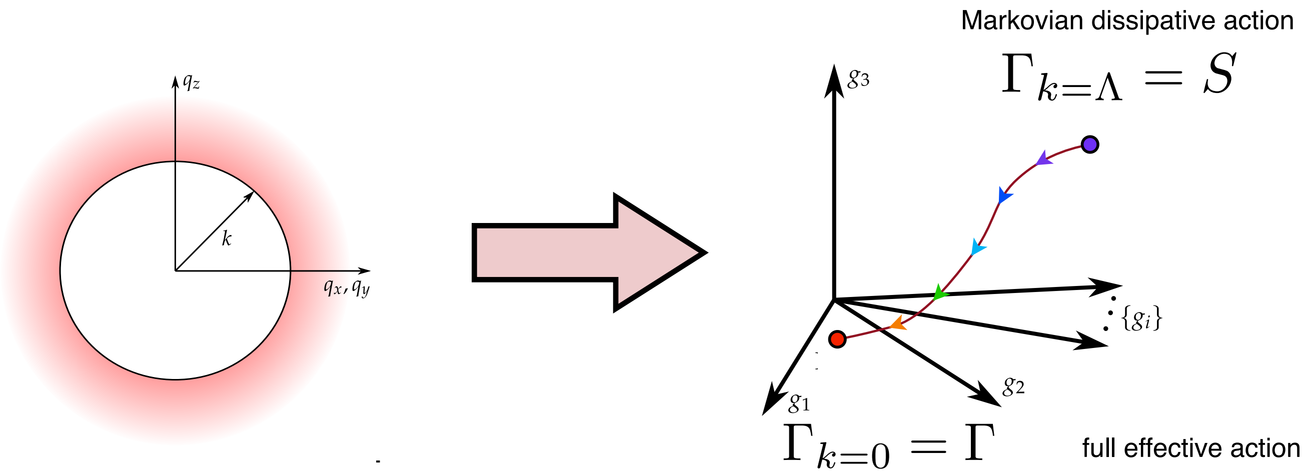
Open System Functional RG

closed system Keldysh:
Schoeller, Meden PRL 07
Gasenzer, Pawlowski, PLB 08;
Berges, Hoffmeister, Nucl. Phys. B, 09

- Evaluation of functional integral via equivalent Functional RG equation adapted to open system

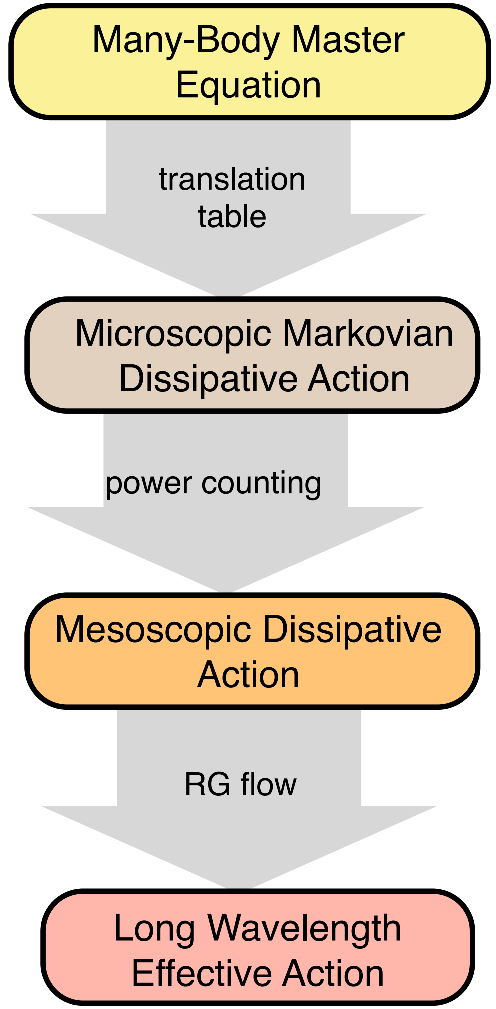
Wetterich, 93

$$\partial_k \Gamma_k = \frac{\mathrm{i}}{2} \mathrm{Tr} \left[\left(\underbrace{\Gamma_k^{(2)}}_{\text{second field variation}} + \underbrace{R_k}_{\text{infrared regulator}} \right)^{-1} \partial_k R_k \right]$$



coarse graining in real space = integrating out high modes in momentum space

mode elimination induces RG flow of coupling of effective action



- solve functional differential equation approximately by systematic derivative expansion truncation
- ordering principle is power counting

Truncation

- explicit ansatz

$$\Gamma_k = \int_X \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & iZ\partial_t + K\Delta \\ iZ^*\partial_t + K^*\Delta & 0 \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} - \left(\frac{\partial U}{\partial \phi_c}\phi_q + \frac{\partial U^*}{\partial \phi_c^*}\phi_q^* \right) + i\gamma\phi_q^*\phi_q \right\}$$

- work in d=3

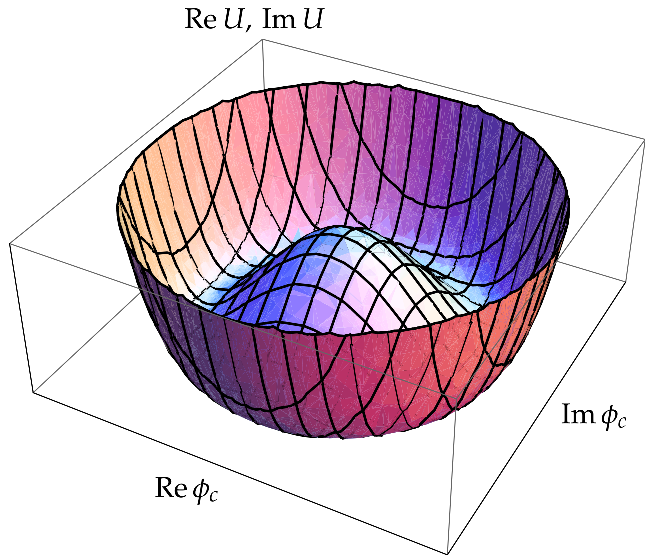
- arbitrary complex running couplings allowed

  - e.g. propagation and diffusion $K = A + \mathrm{i}D$

- includes all non-irrelevant operators (d = 3)

$$U = U(\rho_c) = \frac{u}{2}\left(\rho_c - \rho_0\right)^2 + \frac{u_3}{6}\left(\rho_c - \rho_0\right)^3$$

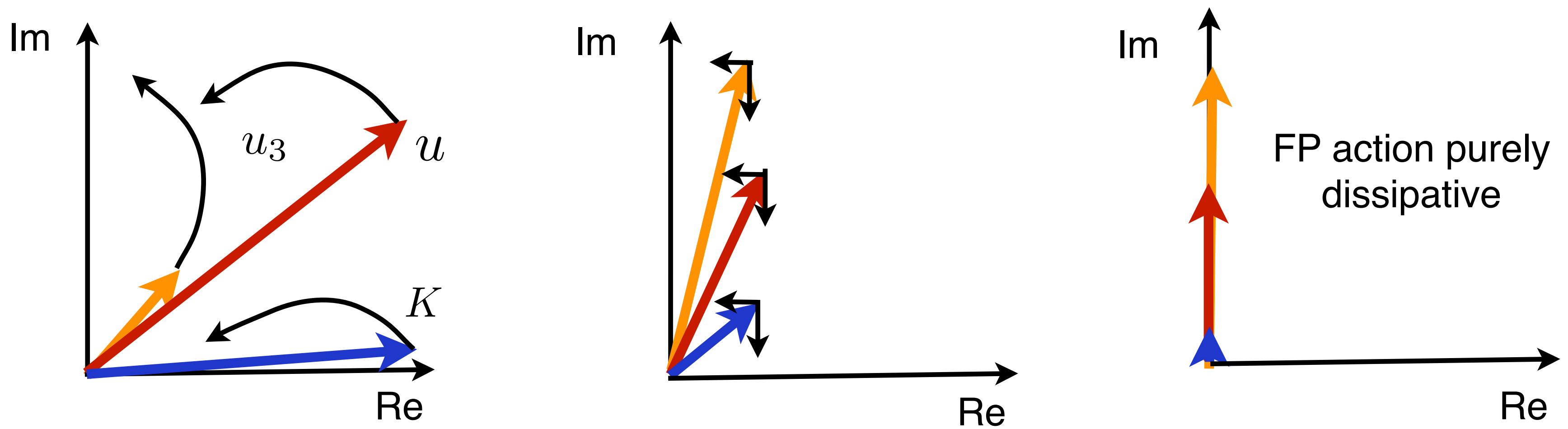
$$\rho_c = \phi_c^*\phi_c$$



- Run the RG <=> follow how these couplings change with scale

Schematic RG flow

- Flow in the complex plane of couplings



non-perturbative initial flow

linearized IR flow

fixed point

- initial values: $\Gamma_{k \approx \Lambda_0} \approx S$
  - particles propagate

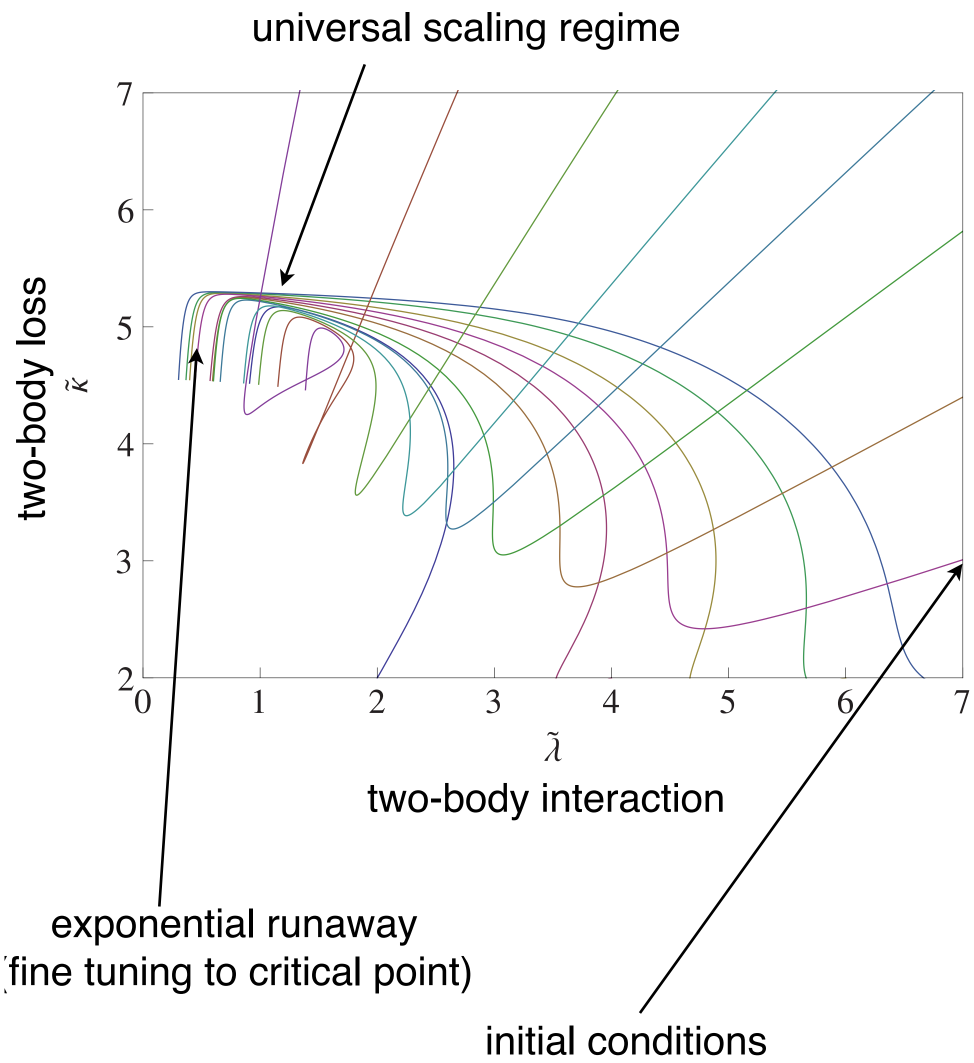
    $A = \mathrm{Re}[K] \approx 1 \gg D = \mathrm{Im}[K]$
  - coherent collisions ~ two-body loss
  - three-body couplings subleading

- universal domain encoding universality class

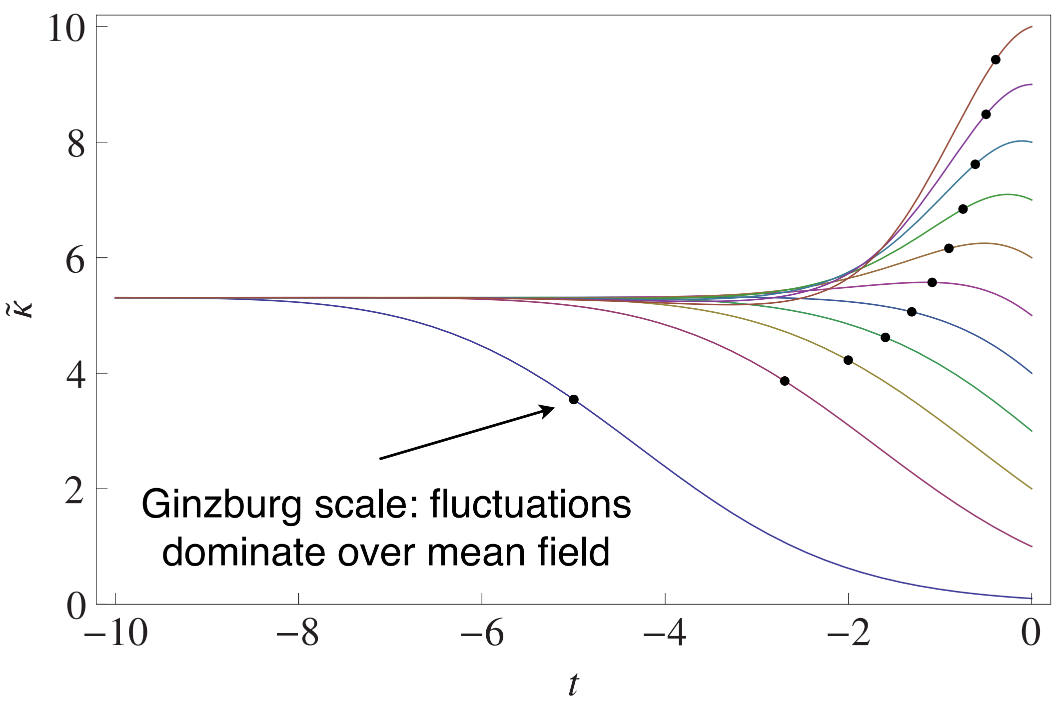
➡ decoherence

Emergence of universality in numerical evaluation

- Flow in the complex plane of couplings

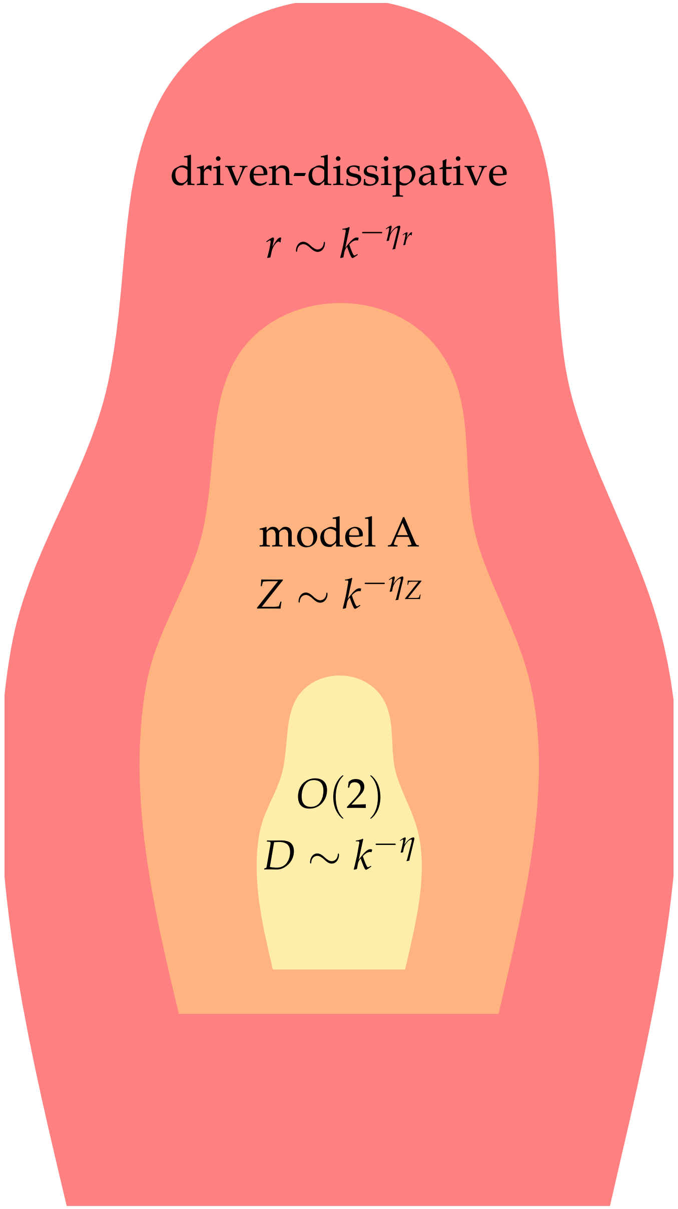


- Extent of universal regime delimited by Ginzburg scale



$$\chi_G \approx \left(\frac{\gamma\kappa}{4\pi D^{3/2}}\right)^2$$

Main Result: Hierarchical Structure of Non-Equilibrium Criticality



- The inner shell:
  - describes static critical exponents

$$\langle \phi^*(r, t=0)\phi(0, t=0)\rangle \sim \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

$$\xi \sim |\tau|^{-\nu}$$

  - result coincides with ab initio equilibrium calculation

$$\eta \approx 0.039$$

$$\nu \approx 0.716$$

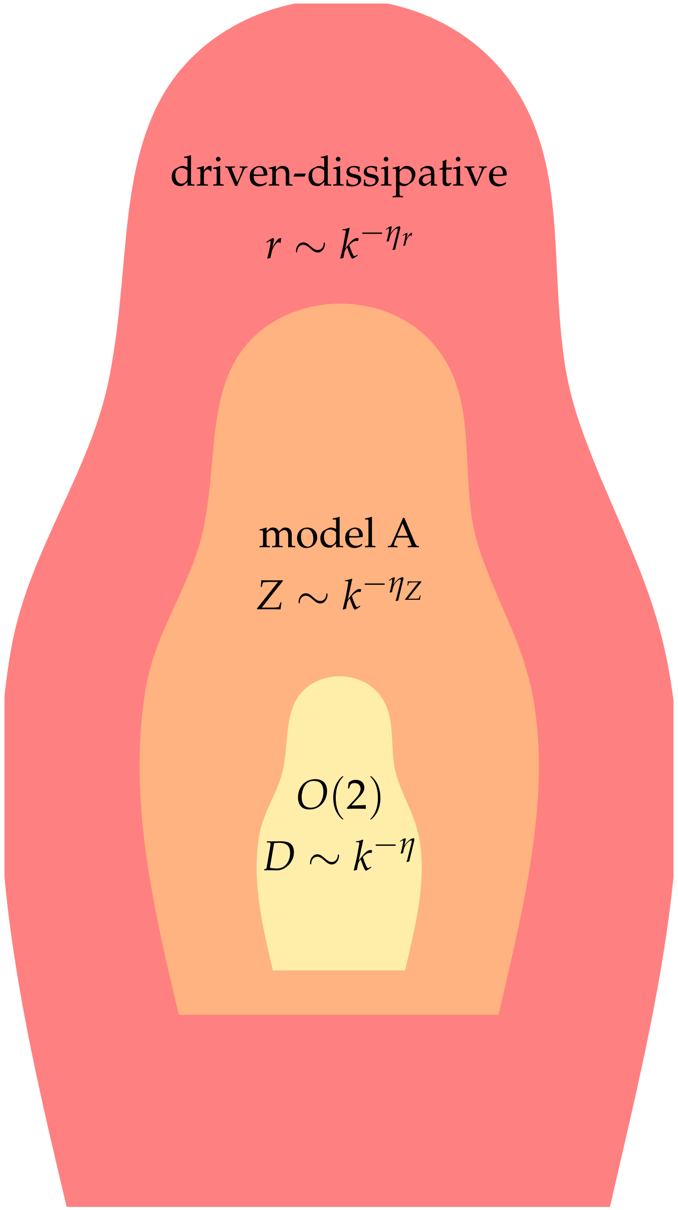
➡ equilibrium exponents of O(2) model unmodified by non-equilibrium condition

➡ quantitative benchmark of our real time approach

cf. Guida and Zinn Justin, J. Phys A (1998)
5 loop order epsilon expansion

$$\eta \approx 0.038(4)$$

Main Result: Hierarchical Structure of Non-Equilibrium Criticality



- The intermediate shell:
  - describes dynamic critical exponent

$$\langle \phi^*(r=0,t)\phi(r=0,0)\rangle \sim \frac{1}{t^{(d-2+\eta_Z)/2}}$$

Info

- introduced in the theory of dynamical critical phenomena (Model A - F) Hohenberg and Halperin, RMP 76
- relaxation to thdyn. equilibrium built in

- result coincides with ab initio Model A calculation

$$\eta_Z \approx 0.161$$

➡ also dynamical exponent of Model A unmodified by non-equilibrium condition

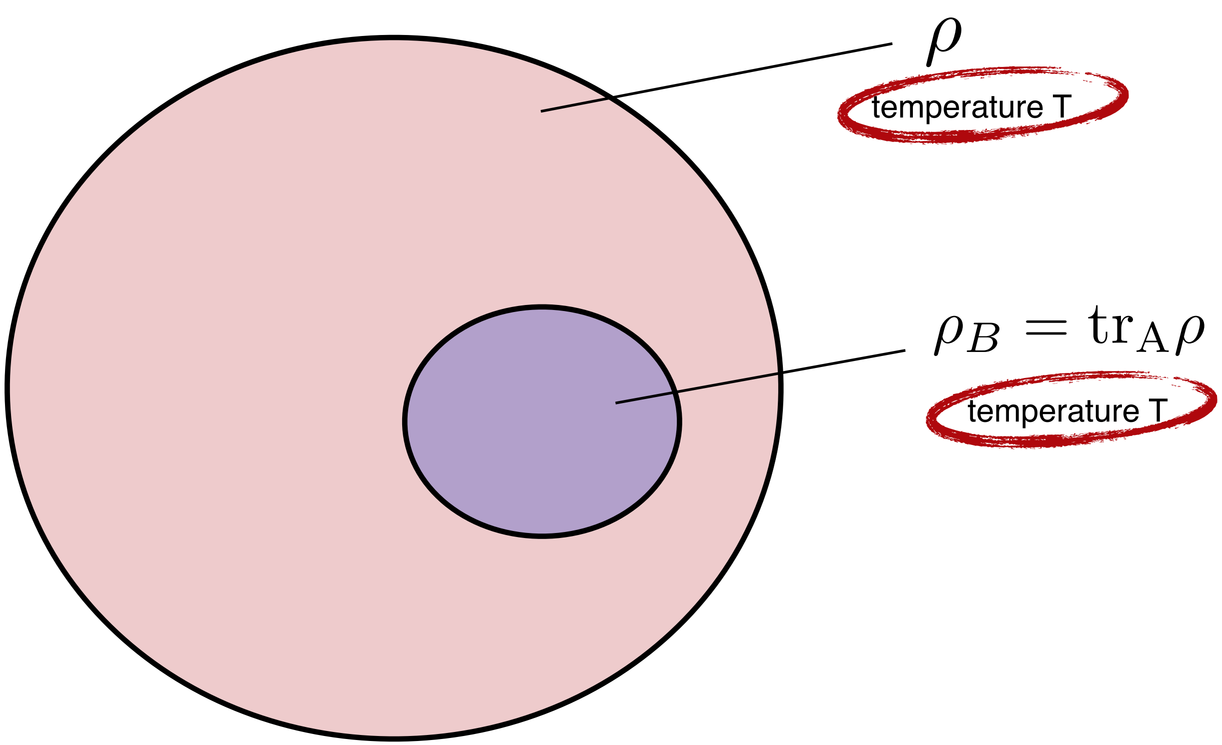
Asymptotic Low-Frequency Thermalization

- dynamic exponent coincides with equilibrium dynamical Model A
- stronger result: asymptotic thermalization of driven-dissipative system

---

- global thermal equilibrium: all subparts in equilibrium with each other

<=> Temperature is invariant under the partition



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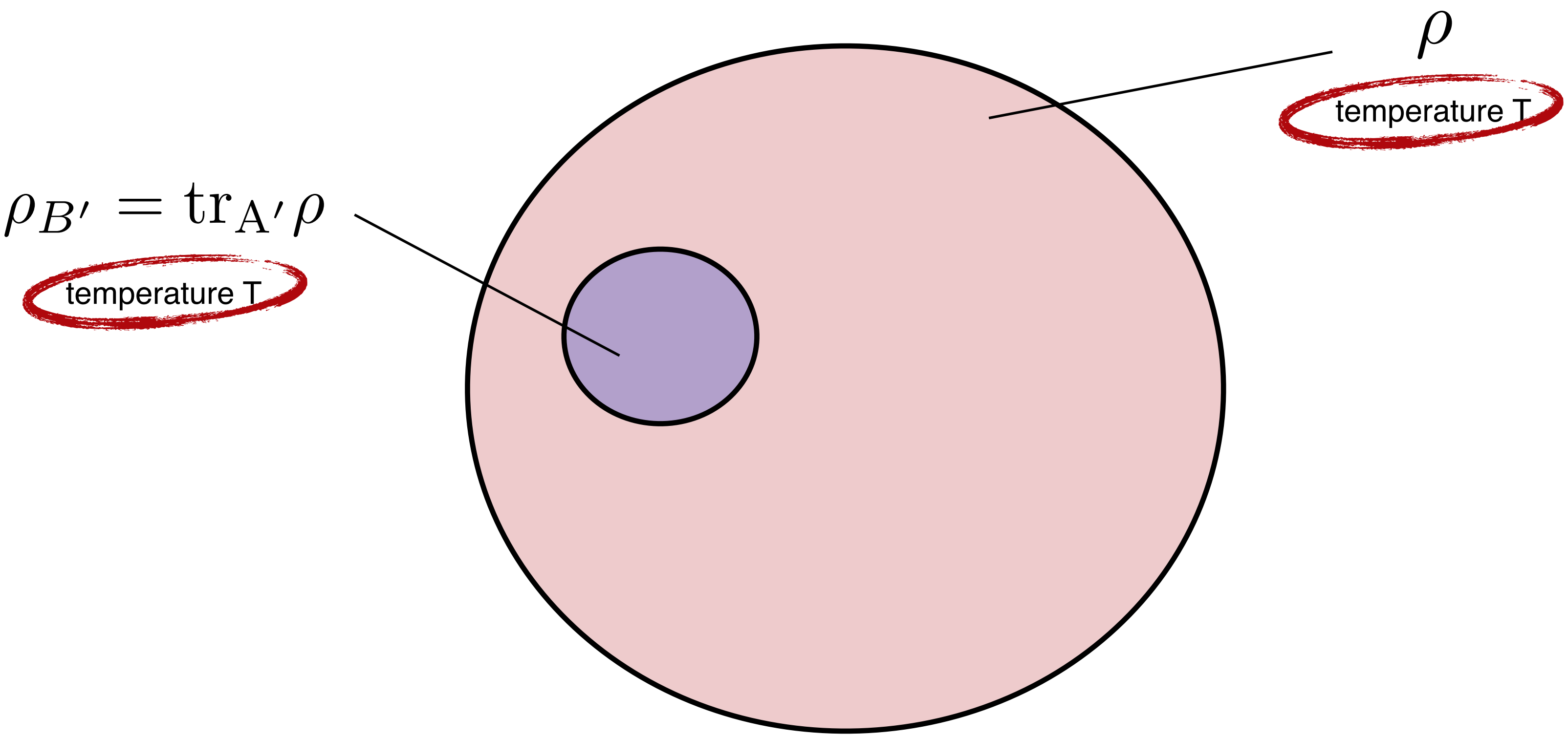

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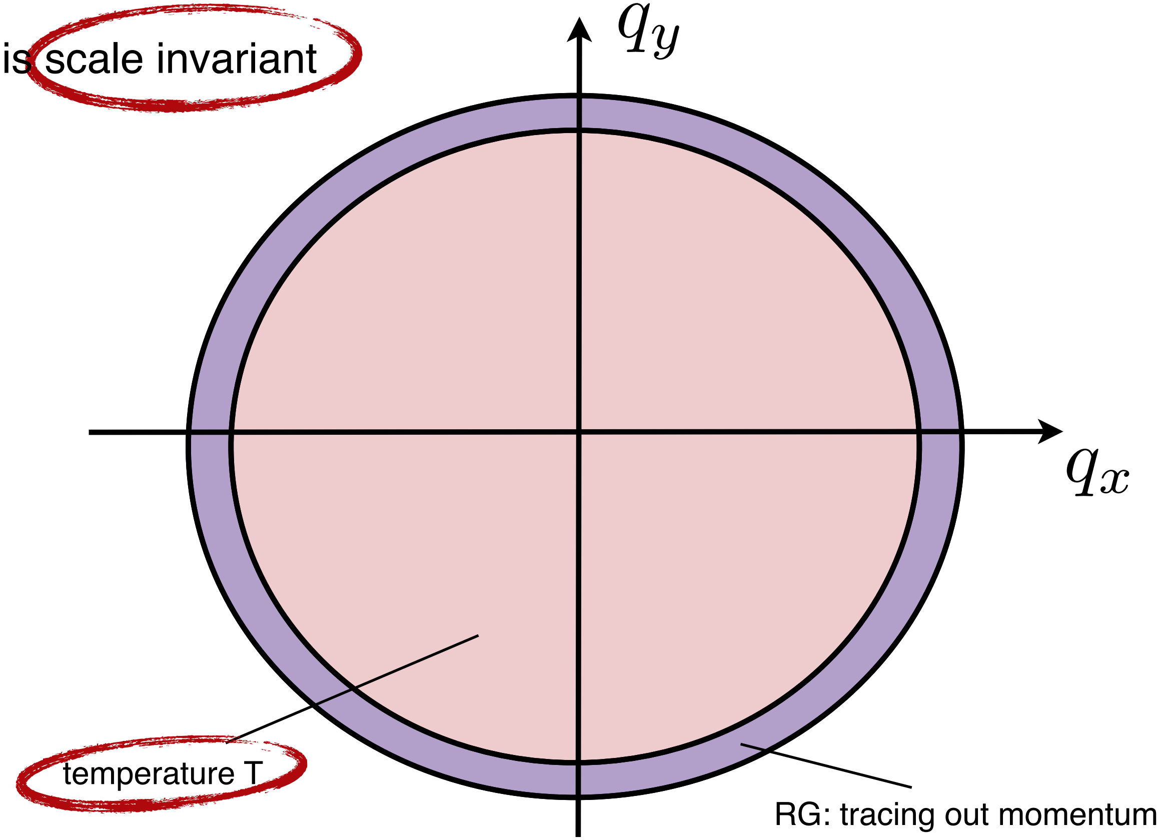
RG: <=> Temperature is scale invariant



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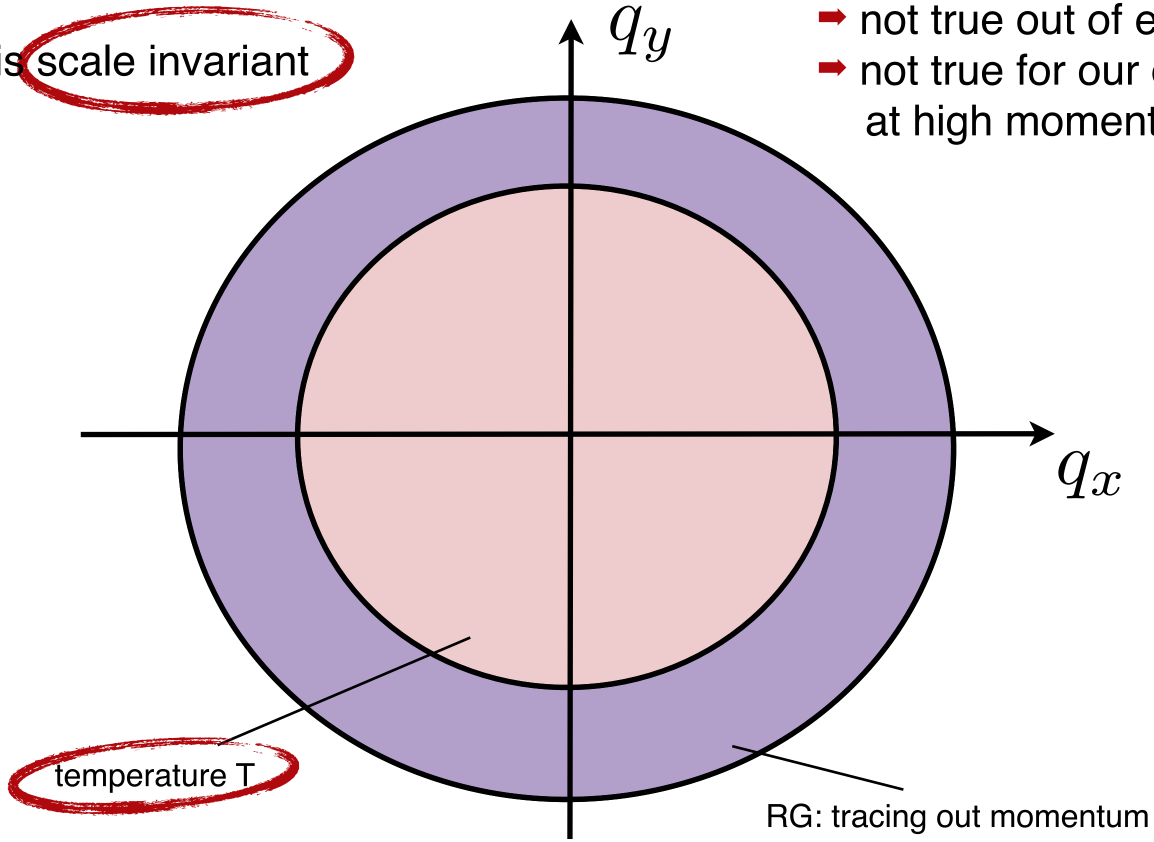
- global thermal equilibrium: all subparts in equilibrium with each other

<=> Temperature is invariant under the partition

RG: <=> Temperature is scale invariant

➡ not true out of equilibrium
➡ not true for our driven-dissipative system at high momenta



Asymptotic Low-Frequency Thermalization

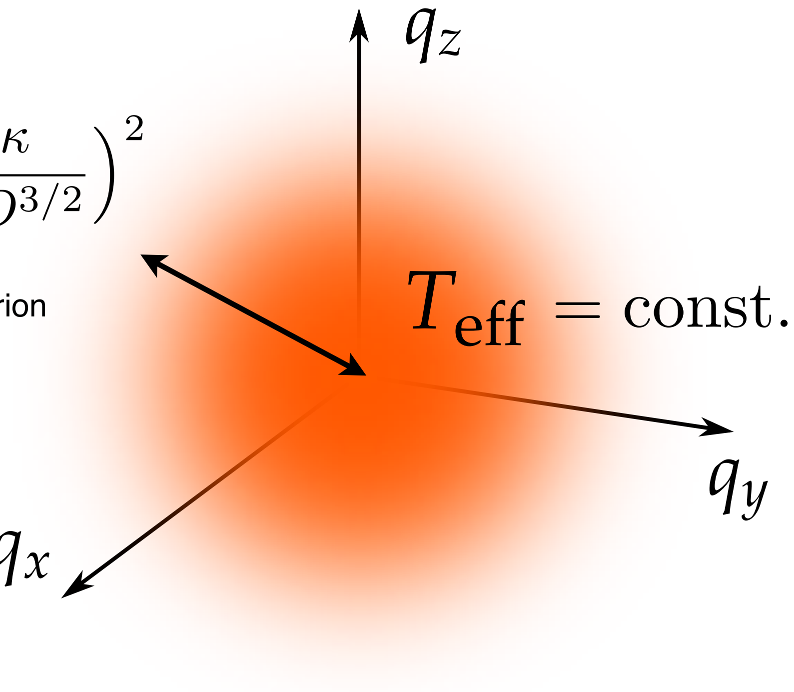
- dynamic exponent coincides with equilibrium dynamical Model A
- stronger result: asymptotic thermalization of driven-dissipative system

---

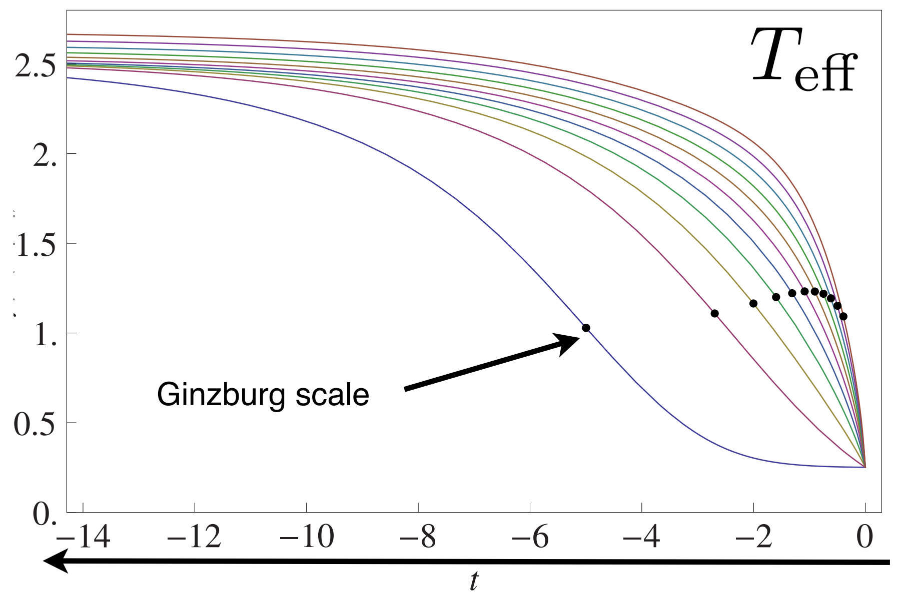
- we find a scale invariant effective temperature in the universal low-momentum regime: asymptotic thermalization

$$\chi_G \approx \left(\frac{\gamma\kappa}{4\pi D^{3/2}}\right)^2$$

Ginzburg criterion



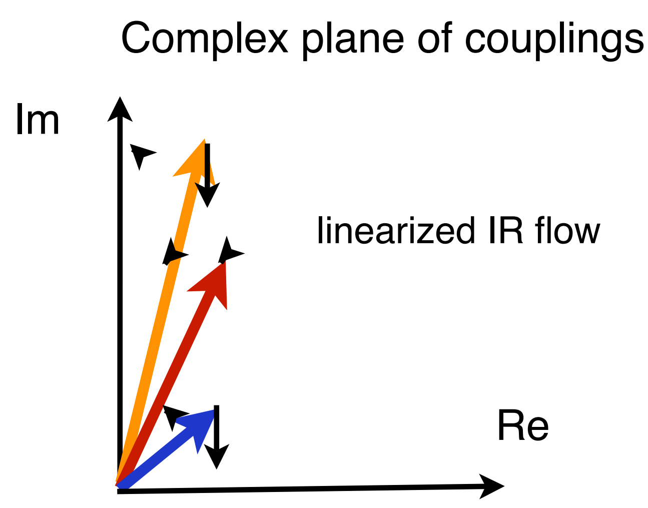
numerical evaluation



flow to long wavelength/small momenta

NB: must be constant in equilibrium

Thermalization: Formal reason



- IR flow of noise and dynamical couplings locked

$$\boxed{\eta_Z(\mathbf{g}_*) = \eta_{\bar{\gamma}}(\mathbf{g}_*)}$$

$$\Gamma_{k\to 0} = \int_X \{\phi_c^* \, iZ\partial_t\phi_q + c.c. + i\bar{\gamma}\phi_q^*\phi_q\} + \ldots$$

$$Z \sim k^{\eta_Z}, \quad \gamma \sim k^{\eta_{\bar{\gamma}}}$$

- emergent "equilibrium" symmetry of $\mathrm{i}\Gamma_{k\to 0}$

Aron et al., J. Stat. Mech. (2010); adapted to real time functional integral

$$\Phi_c(t,\mathbf{x}) \to \Phi_c(-t,\mathbf{x})$$

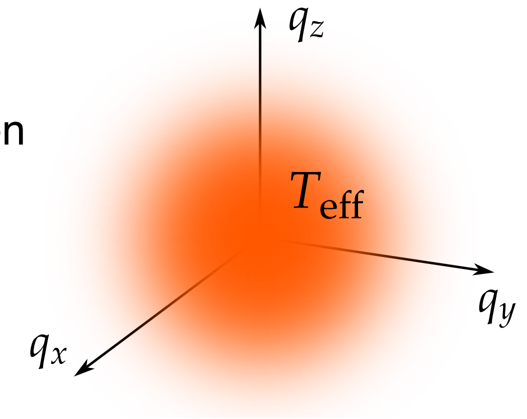
$$\Phi_q(t,\mathbf{x}) \to \Phi_q(-t,\mathbf{x}) + \frac{2|Z^2|}{\bar{\gamma}}\sigma_z\partial_t\Phi_c(-t,\mathbf{x}) \qquad \mathrm{i} \to -\mathrm{i}$$

- interpretation: (Time reversal) o (Time translations)

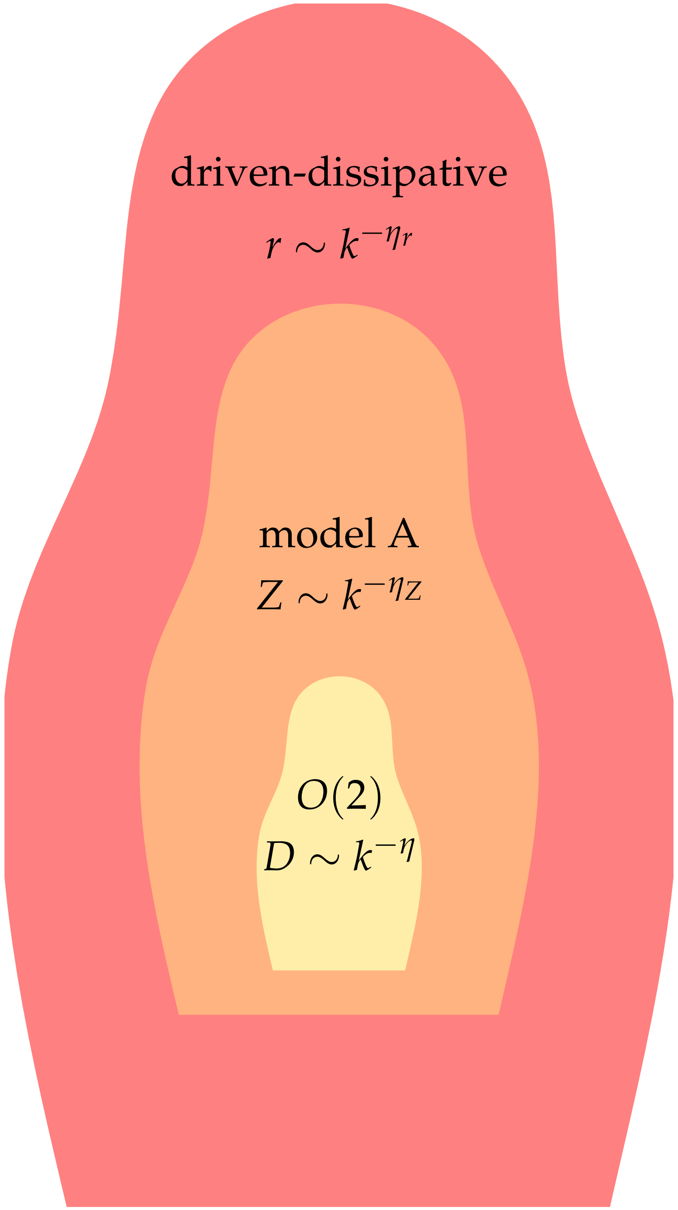
- associated Ward identity implies classical FDT with distribution function

$$F = \frac{2T_{\text{eff}}}{\omega} \qquad T_{\text{eff}} = \frac{\bar{\gamma}}{4|Z|}$$

effective temperature



Main Result: Hierarchical Structure of Non-Equilibrium Criticality



- The outer shell:
  - describes fadeout of coherent vs. dissipative dynamics: universal decoherence

e.g.

coherent propagation / diffusion: $$\frac{A}{D} \sim k^{-\eta_r}$$

two-body elastic collision / two-body loss: $$\frac{\lambda}{\kappa} \sim k^{-\eta_r}$$

- we find:

$$\boxed{\eta_r \approx -0.101}$$

- we show:

➡ exponent is new and independent of the others

➡ the extensions is maximal (no more independent exponents will be found)

➡ defines a new non-equilibrium universality class

Independence of drive exponent

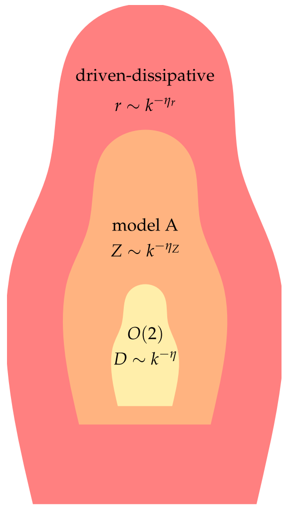
- Argument 1: Infrared
- block diagonal form of linearized flow near fixed point

$$\partial_t \begin{pmatrix} r_K \\ r_u \\ r_{u_3} \\ 1/Z \\ 1/\bar{K} \\ \Delta w \\ \Delta\tilde{\kappa} \\ \Delta\tilde{\kappa}_3 \end{pmatrix} = \begin{pmatrix} 0.0525 & 0.0586 & 0.0317 & 0 & 0 & 0 & 0 & 0 \\ -0.0002 & -0.0526 & 0.1956 & 0 & 0 & 0 & 0 & 0 \\ 0.4976 & -2.3273 & 1.9725 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1605 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0392 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.6204 & 0.0881 & 0.0046 \\ 0 & 0 & 0 & 0 & 0 & -3.1828 & 0.2899 & 0.0363 \\ 0 & 0 & 0 & 0 & 0 & -15.3743 & -42.2487 & 2.1828 \end{pmatrix} \begin{pmatrix} r_K \\ r_u \\ r_{u_3} \\ 1/Z \\ 1/\bar{K} \\ \Delta w \\ \Delta\tilde{\kappa} \\ \Delta\tilde{\kappa}_3 \end{pmatrix}$$

non-equilibrium relaxational equilibrium O(2)

- 4 independent eigenvalues
- structure protected “diagrammatically” (d = 3)

mass exponent	ν
anomalous dimension	η
dynamical exponent	η_Z
drive exponent	η_r



Independence of drive exponent, maximal extension

- Argument 2: Ultraviolet
  - the origin of each independent exponent must be associated to an UV scale

e.g. $$\langle \phi^*(r)\phi(0)\rangle \sim \underbrace{L^{2-d}}_{\text{physical length dimension}} \sim \underbrace{a^{-\eta} r^{2-d+\eta}}_{\text{experimentally observed scaling}}$$

  - counting UV scales: mass matrix and source terms

$$\Gamma = \int_X (\bar\phi_c^*, \bar\phi_q^*) \begin{pmatrix} 0 & -\mu_{UV} + i\chi_{UV} \\ -\mu_{UV} - i\chi_{UV} & \gamma_{UV} \end{pmatrix} \begin{pmatrix} \bar\phi_c \\ \bar\phi_q \end{pmatrix} + f(j_c^* \bar\phi_q + j_q^* \bar\phi_c + c.c.) + \dots$$

- classical O(2) model: (imaginary) mass term, real source term: 2
- Model A: plus Keldysh mass term (temperature): + 1
- driven model: plus real mass term: + 1

4 independent exponents

➡ For N = 2 field components, there cannot be more independent critical exponents

➡ Extension of equilibrium criticality is maximal

Non-equilibrium universality class

- What is the most general microscopic dynamics compatible with stationary Gibbs ensemble?

$$\partial_t \Phi = [-\mathbf{1} + \mathrm{i} R \sigma_z] \frac{\delta \mathcal{H}[\Phi]}{\delta \Phi^\dagger} + \zeta \qquad \langle \zeta^*(t, \mathbf{x}) \zeta(t', \mathbf{x}') \rangle = 2T \delta(t - t') \delta(\mathbf{x} - \mathbf{x}')$$

$$R \geq 0$$

- Proof 1: Stochastic equation of motion: mapping to Fokker-Planck equation, construct stationary solution (Graham 73)

- Proof 2 (symmetry):
    - Use equivalence of stochastic PDE to a functional integral (MSR construction)

$$Z = \int \mathcal{D}(\Phi_q, \Phi_c) \exp \mathrm{i} \int_X \Phi_q^\dagger \left[\mathrm{i}\sigma_z \left(\partial_t \Phi_c + (\mathbf{1} - \mathrm{i} R \sigma_z) \frac{\delta \mathcal{H}[\Phi_c]}{\delta \Phi_c^\dagger} \right) + \mathrm{i} T \Phi_q^\dagger \Phi_q \right]$$

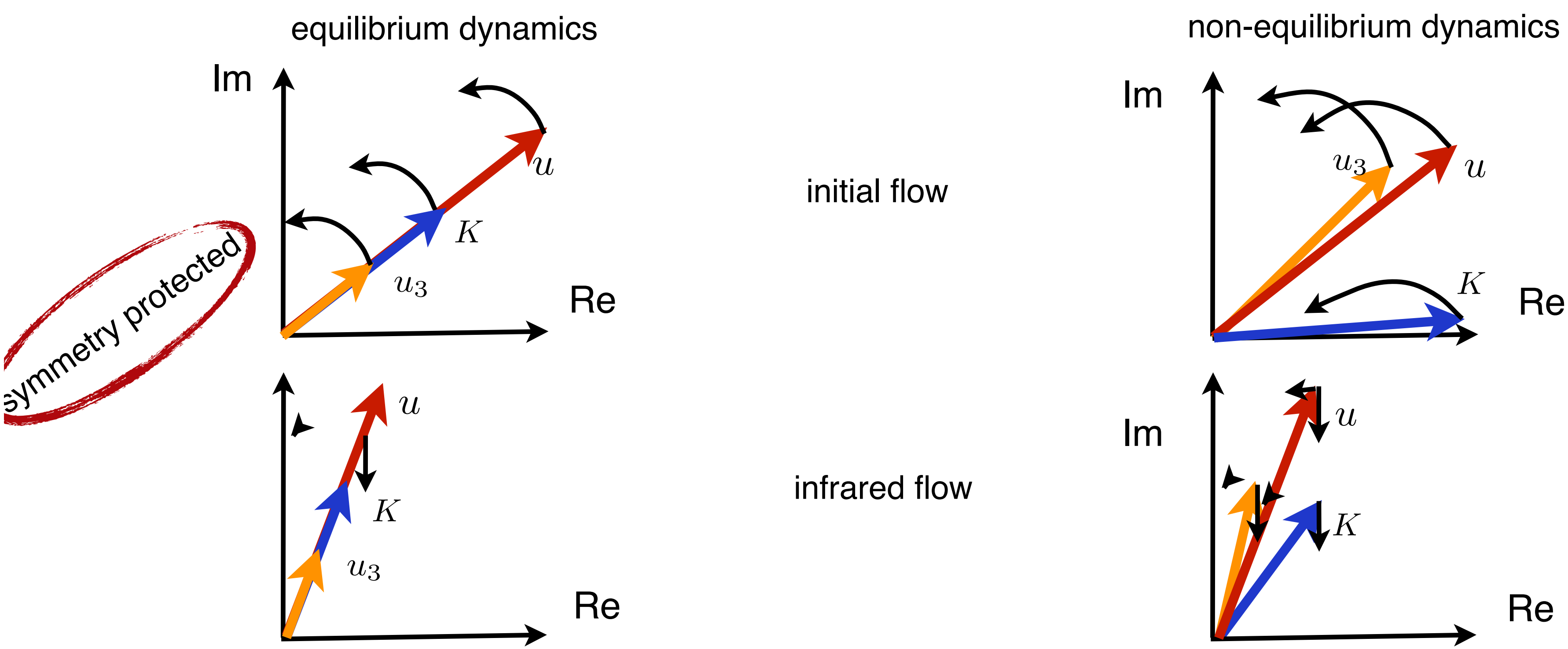
- Check: equilibrium symmetry still present for compatible dynamics
- associated Ward identity implies classical Fluctuation-Dissipation theorem

- Variant 2 allows comparison with driven case:

➡ Equilibrium symmetry absent in general non-equilibrium case

Non-equilibrium universality class

- global thermal equilibrium is ensured by equilibrium symmetry:



- eigenvalue of flow speed

$$\eta_R \approx -0.143$$

- lowest eigenvalue

$$\eta_r \approx -0.101$$

➡ equilibrium and driven systems are in different universality classes

➡ physical reason: independence of coherent and dissipative dynamics

➡ formal reason: difference in symmetry

Observable consequences of driven criticality

- experiments probing the dynamical single-particle renormalized response:

$$\eta_D = \eta_{\bar{D}} - \eta_Z$$

$$G^R(\omega, \mathbf{q}) = \frac{1}{\omega - A_0 |\mathbf{q}|^{2-\eta_r-\eta_D} + i D_0 |\mathbf{q}|^{2-\eta_D}}$$

non-universal constants

  - ultracold atoms: RF spectroscopy (Jin group, Nature 08)

$$\omega \approx A_0 |\mathbf{q}|^{2.22} - i D_0 |\mathbf{q}|^{2.12}$$

peak position and width

  - exciton-polariton systems: homodyne detection (Deveaud-Pledran group, PRL 11)

$$\mathrm{Re}\, G^R(\omega, \mathbf{q}), \;\; \mathrm{Im}\, G^R(\omega, \mathbf{q})$$

measured independently

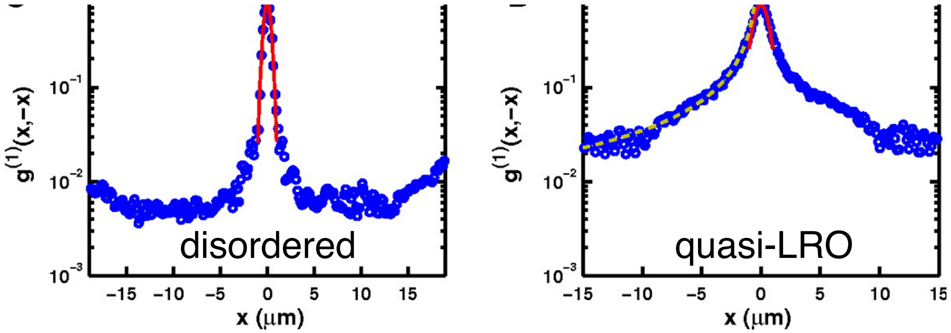
- necessary resolution: extent of critical domain from Ginzburg criterion

  - fluctuation dominated for $\chi_G \approx \left(\dfrac{\gamma \kappa}{4\pi D^{3/2}} \right)^2$

$$D \sim \lambda^2 n^2, \kappa^2 n^2$$

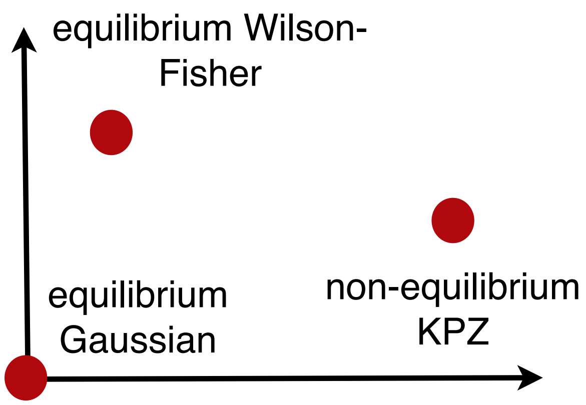
distance from phase transition

Directions



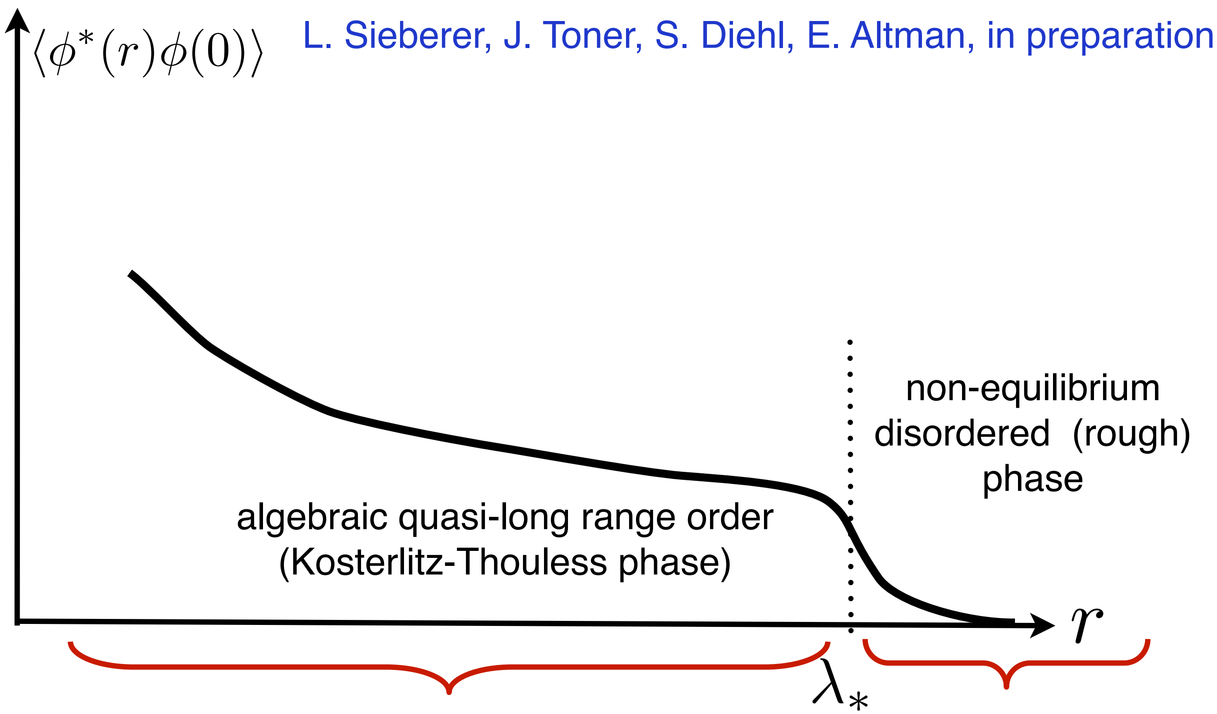
from Roumpos et al., PNAS (2012)

- 2D: exciton-polariton systems as laboratories of nonequilibrium statistical mechanics



- equilibrium fixed point unstable in 2D
- KPZ nixed point relevant

L. Sieberer, J. Toner, S. Diehl, E. Altman, in preparation

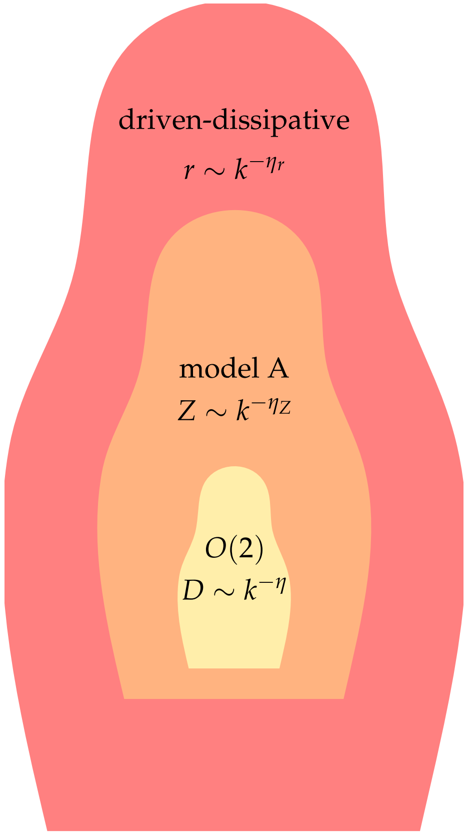


- Quantum criticality in driven open systems with tailored dynamics
- Different symmetries: N = 1: Driven Rydberg ensembles? (Schauss et al., Nature 2012)
- Systems with coherent forcing (Jaynes-Cummings, Nissen et al., PRL 2012)
- Interacting fermionic systems (Eisert, Prosen, 2010, Hoening, Moos, Fleischhauer, PRA 2012)

➡ Classification of universality in driven-dissipative systems

Summary: Universality in driven-dissipative systems

- Hierarchical structure of criticality with no modification of inner shells:
  - static sector
    - classical O(2) model
  - dynamical sector
    - asymptotic low frequency thermalization $\eta_Z = \eta_{\bar{\gamma}}$
    - Halperin-Hohenberg Model A
  - competing unitary and dissipative dynamics
    - universal long-wavelength decoherence
    - measured by an independent critical exponent

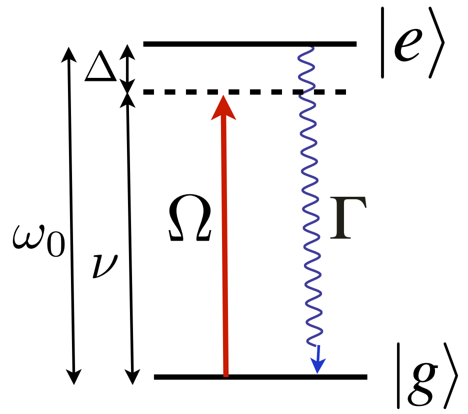


- driven-dissipative systems define new out-of equilibrium universality class
  - independence of coherent/dissipative dynamics
  - different symmetries compared to equilibrium

Open Quantum Systems as Driven Systems

- Most (all?) of the non-equilibrium features to be discussed root in the driven nature of quantum optical systems

- Consider two-level system:
  - without drive, upper level inaccessible
  - drive / pump means to put in large amount of energy. Does not happen “spontaneously”
  - large scale separation: bath may look as zero temperature reservoir though it is not (cf. radiation field)



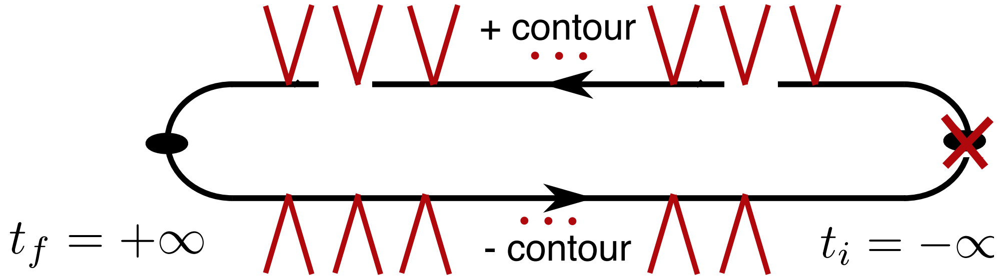
- Implications:
  - no obedience of the second law of thermodynamics (state purification)
  - independent unitary and dissipative dynamics (different physical origins)
  - no guarantee for detailed balance, once unitary and dissipative dynamics compete
  - NB: contrasts equilibrium: relaxational (dissipative) and reversible (coherent) dynamics have the same origin (Hamiltonian)

Part I

Part II

➡ such conditions may be achieved in many-body systems as well (though not generic)

Keldysh functional integral



- real-time partition function:

$$Z = \mathrm{tr}\rho = \mathrm{tr}\rho(t_\mathrm{i}) = 1$$

$$= \mathrm{tr}\,\hat{\mathcal{U}}(t_\mathrm{i}, t_\mathrm{f})\hat{\mathcal{U}}(t_\mathrm{f}, t_\mathrm{i})\rho(t_\mathrm{i}) = \mathrm{tr}\,\hat{\mathcal{U}}(t_\mathrm{f}, t_\mathrm{i})\rho(t_\mathrm{i})\hat{\mathcal{U}}(t_\mathrm{i}, t_\mathrm{f}) = \mathrm{tr}\,\hat{\mathcal{U}}(t_\mathrm{f}, t_\mathrm{i})\rho(t_\mathrm{i})\hat{\mathcal{U}}^\dagger(t_\mathrm{f}, t_\mathrm{i})$$

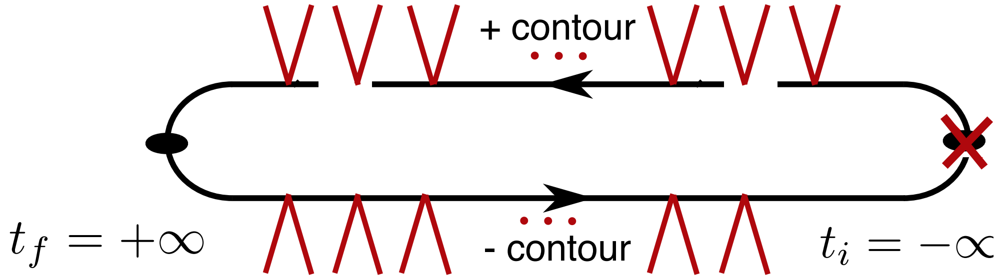
time evolution operator $\hat{\mathcal{U}}(t_\mathrm{f}, t_\mathrm{i}) = e^{-\mathrm{i}H(t_\mathrm{f}-t_\mathrm{i})}$

➡ density operator transforms as matrix under time evolution

➡ Keldysh functional integral: Trotterize on both sides / contours, insert coherent state completeness relations

Refresher

Keldysh functional integral



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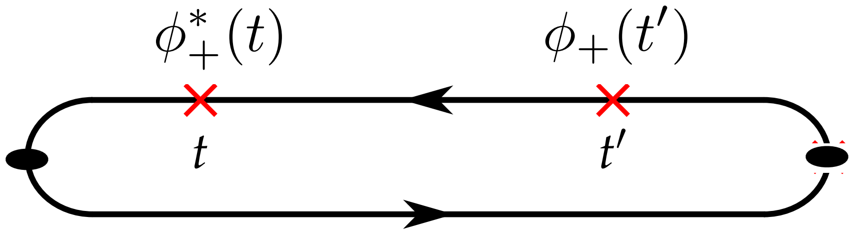
$$= \int \mathcal{D}\phi_+ \mathcal{D}\phi_- \mathrm{e}^{\mathrm{i}S[\phi_+, \phi_-]}$$

Trotterization, coherent state insertion

- correlation functions: field insertions



$$\langle \mathcal{T}_C[\hat{\phi}^\dagger(t)\hat{\phi}(t')]\rangle = \frac{\delta^2 Z[j_+, j_-]}{\delta j_+(t)\delta j^*_+(t')}\Big|_{j=0}$$

$$Z = \langle 1 \rangle$$

$$Z[j_+, j_-] = \langle e^{\mathrm{i}\int (j_+\phi^*_+ + j_-\phi^*_- + c.c.)}\rangle$$

Translation table: Operator vs. Functional Formulation

- Operator formalism: Markovian master equation

$$\partial_t \rho = \mathcal{L}\ \rho = -i\,[H,\rho] + \sum_\alpha \kappa_\alpha \left(2L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}\right)$$ Liouvillian operator

- Functional formalism (equivalent): Markovian dissipative action

$$S = \int_{t_0}^{t_f} dt\ \left(\phi_+^*(t) i\partial_t \phi_+(t) - \phi_-^*(t) i\partial_t \phi_-(t) - i\mathcal{L}(\phi_+^*(t), \phi_+(t), \phi_-^*(t), \phi_-(t))\right).$$

$$\mathcal{L} = -i\,(H_+ - H_-) - \sum_\alpha \kappa_\alpha \left(2L_{\alpha,+}L_{\alpha,-}^\dagger - L_{\alpha,+}^\dagger L_{\alpha,+} - L_{\alpha,-}^\dagger L_{\alpha,-}\right)$$ Liouvillian functional

$$H_\pm = H(\phi_\pm^*, \phi_\pm) \text{ etc.}$$

- ... and partition function

$$Z = \mathrm{tr}\rho(t) = \int \mathcal{D}\,[\Phi_+, \Phi_-]\ \ e^{iS\,[\Phi_+,\Phi_-]} = 1. \qquad \Phi_\pm = (\phi_\pm^*, \phi_\pm)^T$$

- Translation table:

• operator right of density matrix -> - contour
• operator left of density matrix -> + contour